Proof of $a(n)/n$ values for A259748
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Notation example: 47 mod 24 = 23 [remainder operation] vs. $47 \equiv 23 \pmod{24}$ [congruence].
Let n be a positive integer.

$a(n)/n$	0	1/6	1/4	5/12	1/2	2/3	3/4	11/2
n mod 24	1, 2, 5, 7, 10, 11, 13, 17, 19, 23	6	8, 16	12	14, 22	3, 9, 15, 18, 21	4, 20	0

- $\frac{a(n)}{n} = 0 \iff n \equiv 1, 2, 5, 7, 10, 11, 13, 17, 19, 23 \pmod{24} \iff n$ is in A259749.
- $\frac{a(n)}{n} = \frac{1}{6} \iff n \equiv 6 \pmod{24} \iff n$ is in 6*A016813.
- $\frac{a(n)}{n} = \frac{1}{4} \iff n \equiv 8, 16 \pmod{24} \iff n$ is in 8*A001651.
- $\frac{a(n)}{n} = \frac{5}{12} \iff n \equiv 12 \pmod{24} \iff n$ is in A073762 = 12*A005408.
- $\frac{a(n)}{n} = \frac{1}{2} \iff n \equiv 14, 22 \pmod{24} \iff n$ is in 2*A168489.
- $\frac{a(n)}{n} = \frac{2}{3} \iff n \equiv 3, 9, 15, 18, 21 \pmod{24} \iff n$ is in 3*A047584.
- $\frac{a(n)}{n} = \frac{3}{4} \iff n \equiv 4, 20 \pmod{24} \iff n$ is in 4*A007310.
- $\frac{a(n)}{n} = \frac{11}{12} \iff n \equiv 0 \pmod{24} \iff n$ is in A008606 = 12*A001477 = 12*A000027 (for index > 0).

n mod 24	0	1	2	3	4	5	6	7	8	9	10	11
$a(n)/n$	11/12	0	0	2/3	3/4	0	1/6	0	1/4	2/3	0	0
n mod 24	12	13	14	15	16	17	18	19	20	21	22	23
$a(n)/n$	5/12	0	1/2	2/3	1/4	0	2/3	0	3/4	2/3	1/2	0

Proof: Use the formula $a(n) = (3n+2)(n+1)n(n-1)/24 \bmod n$, and consider each possible n modulo 24.

- $n = 24k$:

$$\begin{aligned}
a(24k) &= \frac{(72k+2)(24k+1)(24k)(24k-1)}{24} \bmod 24k \\
&= 2(36k+1)(24k+1)(k)(24k-1) \bmod 24k \\
&= (1728k^3 + 48k^2 - 3k - 1)(24k) + 22k \bmod 24k \\
&= 22k = \frac{11}{12}n.
\end{aligned}$$

- $n = 24k+1$:

$$\begin{aligned}
a(24k+1) &= \frac{(72k+5)(24k+2)(24k+1)(24k)}{24} \bmod 24k+1 \\
&= 2(72k+5)(12k+1)(24k+1)(k) \bmod 24k+1 \\
&= 0.
\end{aligned}$$

- $n = 24k+2$:

$$\begin{aligned}
a(24k+2) &= \frac{(72k+8)(24k+3)(24k+2)(24k+1)}{24} \bmod 24k+2 \\
&= (9k+1)(8k+1)(24k+2)(24k+1) \bmod 24k+2 \\
&= 0.
\end{aligned}$$

- $n = 24k + 3$:

$$\begin{aligned} a(24k+3) &= \frac{(72k+11)(24k+4)(24k+3)(24k+2)}{24} \bmod 24k+3 \\ &= (72k+11)(6k+1)(8k+1)(12k+1) \bmod 24k+3 \\ &= (1728k^3+696k^2+90k+3)(24k+3)+16k+2 \bmod 24k+3 \\ &= 16k+2 = \frac{2}{3}n. \end{aligned}$$

- $n = 24k + 4$:

$$\begin{aligned} a(24k+4) &= \frac{(72k+14)(24k+5)(24k+4)(24k+3)}{24} \bmod 24k+4 \\ &= (36k+7)(24k+5)(6k+1)(8k+1) \bmod 24k+4 \\ &= (1728k^3+912k^2+157k+8)(24k+4)+18k+3 \bmod 24k+4 \\ &= 18k+3 = \frac{3}{4}n. \end{aligned}$$

- $n = 24k + 5$:

$$\begin{aligned} a(24k+5) &= \frac{(72k+17)(24k+6)(24k+5)(24k+4)}{24} \bmod 24k+5 \\ &= (72k+17)(4k+1)(24k+5)(6k+1) \bmod 24k+5 \\ &= 0. \end{aligned}$$

- $n = 24k + 6$:

$$\begin{aligned} a(24k+6) &= \frac{(72k+20)(24k+7)(24k+6)(24k+5)}{24} \bmod 24k+6 \\ &= (18k+5)(24k+7)(4k+1)(24k+5) \bmod 24k+6 \\ &= (1728k^3+1344k^2+345k+29)(24k+6)+4k+1 \bmod 24k+6 \\ &= 4k+1 = \frac{1}{6}n. \end{aligned}$$

- $n = 24k + 7$:

$$\begin{aligned} a(24k+7) &= \frac{(72k+23)(24k+8)(24k+7)(24k+6)}{24} \bmod 24k+7 \\ &= 2(72k+23)(3k+1)(24k+7)(4k+1) \bmod 24k+7 \\ &= 0. \end{aligned}$$

- $n = 24k + 8$:

$$\begin{aligned} a(24k+8) &= \frac{(72k+26)(24k+9)(24k+8)(24k+7)}{24} \bmod 24k+8 \\ &= (36k+13)(8k+3)(3k+1)(24k+7) \bmod 24k+8 \\ &= (1728k^3+1776k^2+605k+68)(24k+8)+6k+2 \bmod 24k+8 \\ &= 6k+2 = \frac{1}{4}n. \end{aligned}$$

- $n = 24k + 9$:

$$\begin{aligned} a(24k+9) &= \frac{(72k+29)(24k+10)(24k+9)(24k+8)}{24} \bmod 24k+9 \\ &= 2(72k+29)(12k+5)(8k+3)(3k+1) \bmod 24k+9 \\ &= (1728k^3+1992k^2+762k+96)(24k+9)+16k+6 \bmod 24k+9 \\ &= 16k+6 = \frac{2}{3}n. \end{aligned}$$

- $n = 24k + 10$:

$$\begin{aligned} a(24k+10) &= \frac{(72k+32)(24k+11)(24k+10)(24k+9)}{24} \bmod 24k+10 \\ &= (9k+4)(24k+11)(24k+10)(8k+3) \bmod 24k+10 \\ &= 0. \end{aligned}$$

- $n = 24k + 11$:

$$\begin{aligned} a(24k+11) &= \frac{(72k+35)(24k+12)(24k+11)(24k+10)}{24} \bmod 24k+11 \\ &= (72k+35)(2k+1)(24k+11)(12k+5) \bmod 24k+11 \\ &= 0. \end{aligned}$$

- $n = 24k + 12$:

$$\begin{aligned} a(24k+12) &= \frac{(72k+38)(24k+13)(24k+12)(24k+11)}{24} \bmod 24k+12 \\ &= (36k+19)(24k+13)(2k+1)(24k+11) \bmod 24k+12 \\ &= (1728k^3+2640k^2+1341k+226)(24k+12)+10k+5 \bmod 24k+12 \\ &= 10k+5 = \frac{5}{12}n. \end{aligned}$$

- $n = 24k + 13$:

$$\begin{aligned} a(24k+13) &= \frac{(72k+41)(24k+14)(24k+13)(24k+12)}{24} \bmod 24k+13 \\ &= (72k+41)(12k+7)(24k+13)(2k+1) \bmod 24k+13 \\ &= 0. \end{aligned}$$

- $n = 24k + 14$:

$$\begin{aligned} a(24k+14) &= \frac{(72k+44)(24k+15)(24k+14)(24k+13)}{24} \bmod 24k+14 \\ &= (18k+11)(8k+5)(12k+7)(24k+13) \bmod 24k+14 \\ &= (1728k^3+3072k^2+1817k+357)(24k+14)+12k+7 \bmod 24k+14 \\ &= 12k+7 = \frac{1}{2}n. \end{aligned}$$

- $n = 24k + 15$:

$$\begin{aligned} a(24k+15) &= \frac{(72k+47)(24k+16)(24k+15)(24k+14)}{24} \bmod 24k+15 \\ &= 2(72k+47)(3k+2)(8k+5)(12k+7) \bmod 24k+15 \\ &= (1728k^3+3288k^2+2082k+438)(24k+15)+16k+10 \bmod 24k+15 \\ &= 16k+10 = \frac{2}{3}n. \end{aligned}$$

- $n = 24k + 16$:

$$\begin{aligned} a(24k+16) &= \frac{(72k+50)(24k+17)(24k+16)(24k+15)}{24} \bmod 24k+16 \\ &= 2(36k+25)(24k+17)(3k+2)(8k+5) \bmod 24k+16 \\ &= (1728k^3+3504k^2+2365k+531)(24k+16)+6k+4 \bmod 24k+16 \\ &= 6k+4 = \frac{1}{4}n. \end{aligned}$$

- $n = 24k + 17$:

$$\begin{aligned} a(24k+17) &= \frac{(72k+53)(24k+18)(24k+17)(24k+16)}{24} \bmod 24k+17 \\ &= 2(72k+53)(4k+3)(24k+17)(3k+2) \bmod 24k+17 \\ &= 0. \end{aligned}$$

- $n = 24k + 18$:

$$\begin{aligned} a(24k+18) &= \frac{(72k+56)(24k+19)(24k+18)(24k+17)}{24} \bmod 24k+18 \\ &= 2(9k+7)(24k+19)(4k+3)(24k+17) \bmod 24k+18 \\ &= (1728k^3+3936k^2+2985k+753)(24k+18)+16k+12 \bmod 24k+18 \\ &= 16k+12 = \frac{2}{3}n. \end{aligned}$$

- $n = 24k + 19$:

$$\begin{aligned} a(24k+19) &= \frac{(72k+59)(24k+20)(24k+19)(24k+18)}{24} \bmod 24k+19 \\ &= (72k+59)(6k+5)(24k+19)(4k+3) \bmod 24k+19 \\ &= 0. \end{aligned}$$

- $n = 24k + 20$:

$$\begin{aligned} a(24k+20) &= \frac{(72k+62)(24k+21)(24k+20)(24k+19)}{24} \bmod 24k+20 \\ &= (36k+31)(8k+7)(6k+5)(24k+19) \bmod 24k+20 \\ &= (1728k^3+4368k^2+3677k+1030)(24k+20)+18k+15 \bmod 24k+20 \\ &= 18k+15 = \frac{3}{4}n. \end{aligned}$$

- $n = 24k + 21$:

$$\begin{aligned} a(24k+21) &= \frac{(72k+65)(24k+22)(24k+21)(24k+20)}{24} \bmod 24k+21 \\ &= (72k+65)(12k+11)(8k+7)(6k+5) \bmod 24k+21 \\ &= (1728k^3+4584k^2+4050k+1191)(24k+21)+16k+14 \bmod 24k+21 \\ &= 16k+14 = \frac{2}{3}n. \end{aligned}$$

- $n = 24k + 22$:

$$\begin{aligned} a(24k+22) &= \frac{(72k+68)(24k+23)(24k+22)(24k+21)}{24} \bmod 24k+22 \\ &= (18k+17)(24k+23)(12k+11)(8k+7) \bmod 24k+22 \\ &= (1728k^3+4800k^2+4441k+1368)(24k+22)+12k+11 \bmod 24k+22 \\ &= 12k+11 = \frac{1}{2}n. \end{aligned}$$

- $n = 24k + 23$:

$$\begin{aligned} a(24k+23) &= \frac{(72k+71)(24k+24)(24k+23)(24k+22)}{24} \bmod 24k+23 \\ &= 2(72k+71)(k+1)(24k+23)(12k+11) \bmod 24k+23 \\ &= 0. \end{aligned}$$

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