Demonstration of formulas

We insert in Excel sequences of "Fifth partial sums of m-th powers”, arranging them in a table as follows:

Fifth partial sums of m-th powers								
n	m=0	m=1	m=2	m=3	m=4	m=5	m=6	m=7
1	1	1	1	1	1	1	1	1
2	6	7	9	13	21	37	69	133
3	21	28	44	82	176	418	1064	2842
4	56	84	156	354	936	2754	8736	29274
5	126	210	450	1200	3750	13080	49350	197400
6	252	462	1122	3432	12342	49632	216342	1001952
7	462	924	2508	8646	35112	159654	787968	4137966
8	792	1716	5148	19734	89232	452166	2489448	14597934
9	1287	3003	9867	41613	207207	1157013	7024407	45454773
10	2002	5005	17875	82225	446875	2724865	18074875	127861825
11	3003	8008	30888	153868	906048	5988268	43072848	330540028
12	4368	12376	51272	274924	1743248	12410476	96186272	795609724

Consider the recurrence relation:

$$a_{(n,m)} = 5a_{(n-1,m)} - 10a_{(n-2,m)} + 10a_{(n-3,m)} - 5a_{(n-4,m)} + a_{(n-5,m)} + n^m$$

which is valid, for each n, in every column of the table. We will use this relationship to derive formulas of the sequences that appear in each row of the table.

For $n = 2$ we have:

$$a_{(2,m)} = 5\times 1 - 0 + 0 - 0 + 0 + 2^m =$$

$$= \boxed{2^m + 5}$$

For $n = 3$:

$$a_{(3,m)} = 5(2^m + 5) - 10\times 1 + 0 - 0 + 0 + 3^m =$$

$$= \boxed{5\times 2^m + 3^m + 15}$$

For $n = 4$:

$$a_{(4,m)} = 5(5\times 2^m + 3^m + 15) - 10(2^m + 5) + 10\times 1 + 5\times 0 + 0 + 4^m$$

$$= \boxed{15\times 2^m + 2^{2m} + 5\times 3^m + 35}$$

Continuing we get:

$$a_{(5,m)} = \boxed{35\times 2^m + 5\times 2^{2m} + 5\times 3^{m+1} + 5^m + 70}$$

$$a_{(6,m)} = \boxed{15\times 2^{2m} + 35\times 2^{m+1} + 35\times 3^m + 5^{m+1} + 6^m + 126}$$

$$a_{(7,m)} = \boxed{35\times 2^{2m} + 63\times 2^{m+1} + 70\times 3^m + 3\times 5^{m+1} + 5\times 6^m + 7^m + 210}$$

and so on …..

This inductive process works indefinitely, generating polynomial expressions longer and longer, which in turn generate sequences with terms that magnify more and more rapidly.

Luciano Ancora