DCL-Chemy III: Hyper-Quadratics

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A Synopsis of the Basics as Covered in DCL-Chemy and DCL-Chemy II

The idea of the *dynamic coefficient list (*or *DCL* for short) is that coefficients may assume diverse values over the course of an iterative procedure. In a preceding article, 'DCL-Chemy Transforms Fibonacci-type Sequences to Arrays' (DCL-Chemy), the formula $(c)F_n + (b)F_{n+1} = F_{n+2}$; $F_0 = 0,\ \ F_1 = 1$ is generalized to

$$(\gamma)F_n + (\beta)F_{n+1} = F_{n+2} \ ; \quad F_0 = 0,\ \ F_1 = 1$$

Where β and γ are the lists $\beta = [b_1, b_2\ldots b_i]$ and $\gamma = [c_1, c_2\ldots c_j]$.

A sequence φ_λ (where λ = the *order* of φ = LCM(i,j)) is generated by applying terms in β and γ in order, according to the iteration being performed. So, at the 1st iteration, the initial F_0 and F_1 are multiplied by c_1 and b_1 respectively. At the 2nd iteration, F_1 and F_2 are multiplied by c_2 and b_2; on the 3rd iteration c_3 and b_3 apply, and so on. After λ iterations, the cycle repeats.

That generates the first sequence: to start with $\beta = [b_2, b_3\ldots b_i, b_1]$ and $\gamma = [c_2, c_3\ldots c_i, c_1]$ generates the next. Permuting β and γ cyclically generates λ distinct sequences. In the context of an array, these sequences are aligned vertically and designated as $S_1, S_2, S_3\ldots S_\lambda$. Arrays are typically represented by $\Phi_\lambda\,[\beta][\gamma]$, with β, γ and λ in numerical form.

Define F_n/F_{n-1}, for $n \to \infty$, as a *limit ratio*. As a rule, each sequence in an array converges, simultaneously and in two directions, to λ positive and λ negative limit ratios. Formulas derived in DCL-Chemy operate on the elements of Φ_λ to provide the coefficients of λ different quadratic equations (Q_j) that have roots corresponding to two specific limit ratios. These sets of equations are called *Q-sets*.

In the article DCL-Chemy II: Reflections and Other Symmetries, the order of terms in β and γ was inverted to create complementary Q-sets. Further investigation revealed unexpected crossover connections between the roots of equations in these sets. This paper follows up on the discovery that symmetrical inversions of certain β and γ configurations allow roots from the two sets to combine as points on hyperbolic curves.

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Quadratic Root Reflections on Hyperbolic Curves

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Symmetric Inversions

Definition: A *symmetric* inversion $\Phi_\lambda \to \cancel{\Phi}_\lambda$ is symmetrical with respect to the order of terms in β and γ. That is, it directly inverts/reverses/reflects the sequential order of the terms in each of the lists β and γ.

$$\beta \to \cancel{\beta} = [b_1, b_2 \ldots b_\lambda] \to [b_\lambda, b_{\lambda-1} \ldots b_1] = [\cancel{b}_1, \cancel{b}_2 \ldots \cancel{b}_\lambda] \qquad \gamma \to \cancel{\gamma} = [c_1, c_2 \ldots c_\lambda] \to [\cancel{c}_1, \cancel{c}_2 \ldots \cancel{c}_\lambda]$$

For example:

$\Phi_3[1,2,3][1,2,3]$	S_1	S_2	S_3	$\cancel{\Phi}_3[3,2,1][3,2,1]$	S_1	S_2	S_3
F_{-4}	$^{-14}/_{18}$	$^{-15}/_{6}$	$^{-11}/_{12}$		$^{-14}/_{6}$	$^{-11}/_{18}$	$^{-15}/_{12}$
$F_{-\lambda}$	$^{4}/_{6}$	$^{9}/_{6}$	$^{4}/_{6}$		$^{8}/_{6}$	$^{3}/_{6}$	$^{6}/_{6}$
F_{-2}	$^{-2}/_{6}$	$^{-3}/_{3}$	$^{-1}/_{2}$		$^{-2}/_{2}$	$^{-1}/_{3}$	$^{-3}/_{6}$
F_{-1}	$^{1}/_{3}$	$^{1}/_{1}$	$^{1}/_{2}$		$^{1}/_{1}$	$^{1}/_{3}$	$^{1}/_{2}$
F_0	0	0	0		0	0	0
F_1	1	1	1		1	1	1
F_2	1	2	3		3	2	1
F_λ	4	9	4		8	3	6
F_4	15	11	14		11	15	14
F_5	19	40	54		57	36	20
$F_{2\lambda}$	68	153	68		136	51	102

Table 1: Symmetrical inversion of $\Phi_3[1,2,3][1,2,3] \to \cancel{\Phi}_3[3,2,1][3,2,1]$

An equation that uses terms from these arrays to construct quadratic equations (Q_j) is stated below. The zeros (roots) of these Q_j are the limit ratios to which the quotients F_n/F_{n-1} in table 1 columns converge.

$$Q_j = F_{\lambda,\,j} \cdot x_j^2 - (F_{\lambda+k,\,j} - F_{\lambda-k,\,j+k} \cdot c)\,x_j - F_{\lambda,\,j+k} \cdot c \tag{1.1}$$

Now (1.1) applies to table 1 sequences to create the equations of corresponding color in table 2. We'll call these two sets of three equations, Q-*sets*. (The order of the last two $\cancel{Q}_j$ in the set to the right has been reversed, so as to have Q_j with identical c coefficients on the same line.)

$Q_1 = 4x_1^2 - 13x_1 - 9$	$\cancel{Q}_1 = 8x_1^2 - 5x_1 - 9$
$Q_2 = 9x_2^2 - 5x_2 - 8$	$\cancel{Q}_2 = 6x_2^2 - 11x_2 - 8$
$Q_3 = 4x_3^2 - 11x_3 - 12$	$\cancel{Q}_3 = 3x_3^2 - 13x_3 - 12$

Table 2: Q_j and $\cancel{Q}_j$ coefficients as derived from the columns in table 1

Note that $b + \cancel{b} = c + \cancel{c}$. In preparation for a demonstration of how roots of these equations combine as points on the hyperbolic curve in figure 1, below, it is instructive to first examine a simple case.

Consider the equation $Q = ax^2 + bx + c$. Let $a = 1$ and $b = c$. Then two formulas that equate the roots of this equation, r_+ and r_-, to its coefficients will combine as below to create a third formula:

$$b = c \qquad \text{i) } r_+ + r_- = -b \qquad \text{ii) } r_+ \cdot r_- = c \qquad \text{iii) } r_+ \cdot r_- + r_+ + r_- = 0$$

A more familiar rendition of iii is

$$xy + x + y = 0 \tag{1.2}$$

Thus the roots of, say, $Q_\phi = x^2 - x - 1$ will combine to identify two points, represented by large black dots on the graph of (1.2), in figure 1 below. (Roots equate in turn to both x and y; hence, two points per pair.)

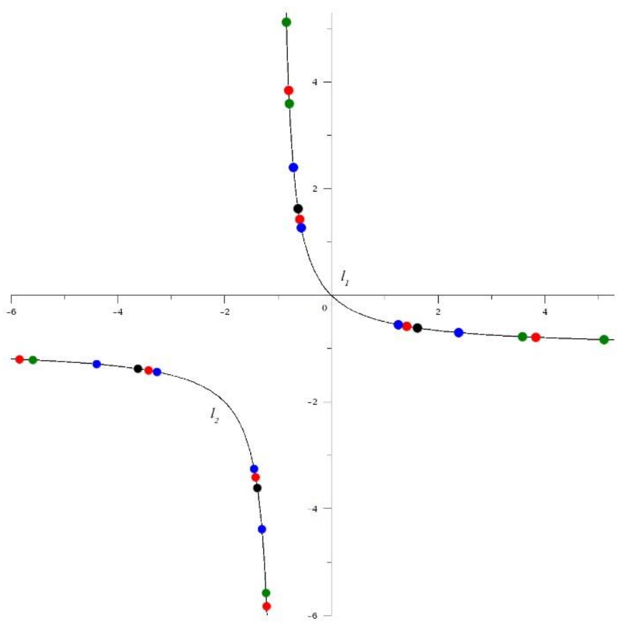

Figure 1: Quadratic roots combine as points on the hyperbola $xy + x + y = 0$

The graph of (1.1) is hyperbolic, comprising curved lines that are mirror images. They are symmetrical with respect to two diagonals, one that passes through the origin (0,0) and a perpendicular that intersects that at (–1,–1). Call the curved line that transits the origin l_1 and the other l_2. Of the various points that are marked on l_1, those black in color are, as stated, the roots of Q_ϕ. We'll see now how the others are derived.

When $b = c$, a quadratic's real roots identify points on l_1 or l_2 of (1.2). For $\beta = \gamma$, $\lambda > 1$, it seems that a *pair* of equations is required, and a root from each, taken in combination, identifies a point on the hyperbola. With $\beta = \gamma$ and $\lambda = 3$ in table 1, the table 2 equations must each contribute a root for a point on (1.2).

For example, take the equations atop table 2: Q_1 and $\cancel{Q}_1$. $4x^2 - 13x - 9$ has roots $r_{1+} = 3.8365$, $r_{1-} = -0.5865$; $8x^2 - 5x - 9$ has roots $\cancel{r}_{1+} = 1.4182$, $\cancel{r}_{1-} = -0.7932$. Then, in (1.2), $3.8365(-0.7932) + 3.8365 - 0.7932 = 1.4182(-0.5865) + 1.4182 - 0.5865 = 0$. These numbers locate four large red dots on l_1 in figure 1.

A symmetry identified earlier in the graph of (1.1) is now used to map points from l_1 to l_2. Since $x = y$ at both (0,0) and (–2,–2), this entails adding 2 to each root and reversing its sign. Thus to map the black dots to l_2, take $(1.618 + 2)(-1) = -3.618$ and $(-0.618 + 2)(-1) = -1.382$. Multiplying $(x + 3.618)(x + 1.382)$ gives $x^2 + 5x + 5$, the roots of which locate the two, smaller black dots on l_2 in figure 1.

Adapted, this procedure also maps the colored dots to l_2 and gives the equations below.

$$
\begin{aligned}
Q_1' &= 4x_1^2 + 29x_1 + 33 & \qquad \cancel{Q}_1' &= 8x_1^2 + 37x_1 + 33 \\
Q_2' &= 9x_2^2 + 41x_2 + 38 & \qquad \cancel{Q}_2' &= 6x_2^2 + 35x_2 + 38 \\
Q_3' &= 4x_3^2 + 27x_3 + 26 & \qquad \cancel{Q}_3' &= 3x_3^2 + 25x_3 + 26
\end{aligned}
$$

Table 3: Quadratics built on roots reflected to l_2

The table 3 equations of the same color interact in the same way as those in table 2. That is, an r_+ (i.e., the $+\sqrt{b^2 - 4ac}$) root will pair with the r_- (the $-\sqrt{b^2 - 4ac}$) root of its complementary equation to define a point (small dot) on the line l_2. Based on the structure of this mapping algorithm, it seems appropriate to identify equations with roots on l_1 as *primary*, and those with roots on l_2 as *secondary* or *reflected*.

Certain attributes of the primary Q_j are unchanged by mapping l_1 roots to l_2; e.g., the a coefficients, their shared discriminant (D) and so forth. Reasons for this will later become clear.

The procedure so far: DCLs generalize a φ-sequence formula to generate sets of sequences that we align in an array. When $\beta = \gamma$ (within not-yet-entirely-defined constraints), a symmetrical inversion of their terms gives another array related in a particular way. Then formula (1.1) applies to find coefficients of equations that have roots to which the ratios of adjacent terms in the sequences converge. These equations form the sets Q_j and $\cancel{Q}_i$, roots of which combine as points on a hyperbolic line. Then, as just explicated, the roots on l_1 can be mapped to l_2, and coefficients for two new sets of equations derived from that.

In sum, these procedures require us to: 1) construct arrays Φ_λ and $\cancel{\Phi}_\lambda$; 2) use (1.1) to derive coefficients for the primary equations Q_j and $\cancel{Q}_i$; 3) map these roots to l_2; 4) use these l_2 roots to find coefficients for two sets of secondary equations. All in all, a bit of labor… Can a more direct method be found?

Quadratic Coefficient Matrices

Remarkably, a shortcut exists that ultimately obviates all of this effort. It begins with configuring Q-set coefficients as matrices, and deriving a set of transform matrices from that.

To derive the first transform matrix, let coefficients of table 2 equations be arrayed in two 3x3 matrices M and $\cancel{M}$ as below. The operation $M^{-1} \cdot \cancel{M} = \cancel{M}^{-1} \cdot M$ produces, as a 'quotient', a matrix (T_1) that transforms one to the other: i.e., $M \cdot T_1 = \cancel{M}$ and $\cancel{M} \cdot T_1 = M$. Next, the matrix M' is fashioned from the coefficients of the Q_j' in table 3 (those with roots that identify points on l_2). Then $M^{-1} \cdot M' = T_2$, the transform matrix at the bottom right below:

$$
M^{-1} \cdot \cancel{M} \;=\; \begin{pmatrix} 4 & -13 & -9 \\ 9 & -5 & -8 \\ 4 & -11 & -12 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 8 & -5 & -9 \\ 6 & -11 & -8 \\ 3 & -13 & -12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = T_1
$$

$$
M^{-1} \cdot M' = \begin{pmatrix} 4 & -13 & -9 \\ 9 & -5 & -8 \\ 4 & -11 & -12 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 & 29 & 33 \\ 9 & 41 & 38 \\ 4 & 27 & 26 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = T_2
$$

Table 4: Matrix 'division' derives the transforms T_1 and T_2

Finally, $M^{-1} \cdot \cancel{M}' = T_1 \cdot T_2$ finds the last of the four transforms to complete the group below:

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & 4 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 1 & 4 & 4 \\ -1 & -3 & -2 \\ 1 & 2 & 1 \end{pmatrix}$$

Table 5: The four T-matrices

These matrices are isomorphic by matrix multiplication to the Klein four-group. Henceforth it requires just the construction of one array to find all of the related Q-set coefficients. Moreover, the yet-to-be-determined conditions on β and γ needed to ensure that Φ_λ and $\cancel{\Phi}_\lambda$ equations will have the same discriminant are no longer relevant. This because T_j is, *conformal* to *any* 3-column matrix M (but zeros in M can give strange results). E.g., take the two random coefficient matrices (i and ii) below.

i) $(12 \quad 14 \quad 13) \cdot T_1 = (11 \quad 12 \quad 13)$ ii) $(2 \quad -1 \quad -6) \cdot T_1 = (-3 \quad -11 \quad -6)$

These examples open views to new territory: i.e., the (i) equation $12x^2 + 14x + 13$ has complex (conjugate) roots; $r_+ = -0.5417 + 0.9345i$ and $r_- = -0.5417 - 0.9345i$. As expected, the roots of $11x^2 + 12x + 13$ are complex and conjugate as well; $\cancel{r}_+ = -0.5833 + 0.8620i$ and $\cancel{r}_- = -0.5833 - 0.8620i$. These imaginary components notwithstanding, the complementary roots combine as usual as solutions to $xy + x + y = 0$. (But how can pairs such as these be graphed as points on a line or surface?)

The second (ii) example is unusual in that the roots of $2x^2 - x - 6$ combine with $-3x^2 - 11x - 6$ roots in two ways. That is, the roots 2 and –0.6666 identify a point on l_1, but –3 and –1.5 are points on l_2. The roots of equations that T_2 and T_3 return have the same pattern.

Interesting variants may perhaps be found in other random examples, but generalizing to $Q = ax^2 + bx + \mathrm{c}$ helps clear up some of the mystery.

$$(a \quad b \quad c) \cdot \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = (a - b + c \quad 2c - b \quad c)) \qquad \cancel{Q} = (\mathrm{a} - b + c)x^2 + (2c - b)x + c$$

Figure 2: The mechanics of the T_1 transform

While this clarifies the mechanics of the process, it's still a surprise that a simple shuffle of Q's coefficients creates a $\cancel{Q}$ with such peculiar complimentary properties inherent to its roots. Now even the T_1 transform is no longer needed: just put a, b and c into $(a - b + c)x^2 + (2c - b)x + c = 0$ and we're done.

To verify this process, doing the math shows that the roots of $ax^2 + bx + c$ and $(a - b + c)x^2 + (2c - b)x + c$ zero out in the equation below:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{b - 2c - \sqrt{b^2 - 4ac}}{2(a - b + c)} + \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{b - 2c - \sqrt{b^2 - 4ac}}{2(a - b + c)} = 0$$

Then $(a \;\; b \;\; c) \cdot T_2$ gives $ax^2 + (4a - b)x + 4a - 2b + c$, which has the roots

$$-2+\frac{b\pm\sqrt{b^2-4ac}}{2a}$$

The roots above illustrate perfectly the process used to create them, which was to add 2 to the original root and change its sign. Yet while these four T-matrices simplify things considerably, they apply only to a specific case (i.e., $k = 1$) of the more general version of (1.2) below:

$$kxy + x + y = 0 \tag{1.3}$$

For example, let $k = 2$: then matrices constructed on Φ_3 $[2c_1,2c_2,2c_3][c_1,c_2,c_3]$ coefficients have roots on $2xy + x + y$. The transforms (T') that work for these matrices are juxtaposed above the originals below.

$$T_0' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_1' = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 4 & 4 & 1 \end{pmatrix} \qquad T_2' = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_3' = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & -1 \\ 4 & 4 & 1 \end{pmatrix}$$

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & 4 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 1 & 4 & 4 \\ -1 & -3 & -2 \\ 1 & 2 & 1 \end{pmatrix}$$

Strangely, the T-matrices themselves seem to have undergone transformations, where 1 and 2 have swapped places and all three are rotated by 180º. Generalizing these transforms clears this up. We'll see that T_1 and T_2 have an inverse relationship. One shrinks as the other grows and, together, these matrices model both the curvature of $kxy + x + y$ and the distance between its lines. (I.e., note that the absolute value of both x and y at the apex of l_2 is $2/k$.) Setting k as an upper index, the set of generalized T-matrices that transform a matrix built on coefficients in the set Q_λ derives, empirically, as:

$$T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_1^k = \begin{pmatrix} 1 & 0 & 0 \\ -k & -1 & 0 \\ k^2 & 2k & 1 \end{pmatrix} \qquad T_2^k = \begin{pmatrix} 1 & 4/k & 4/k^2 \\ 0 & -1 & -2/k \\ 0 & 0 & 1 \end{pmatrix} \qquad T_3^k = \begin{pmatrix} 1 & 4/k & 4/k^2 \\ -k & -3 & -2/k \\ k^2 & 2k & 1 \end{pmatrix}$$

Table 6: The generalized transformation matrix set for $kxy + x + y$

As a check, these generalized T-matrices too are isomorphic to the Klein group. For the full equation set, let $M = (a \;\; b \;\; c)$ and multiply it in turn by the T_j^k in table 6.

$$T_0 \;\rightarrow\; ax^2 + bx + c \qquad T_1^k \;\rightarrow\; (ck^2 - bk + a)x^2 + (2ck - b)x + c$$

$$T_2^k \;\rightarrow\; ax^2 + (4a/k - b)x + 4a/k^2 - 2b/k + c$$

$$T_3^k \;\rightarrow\; (ck^2 - bk + a)x^2 + (4a/k - 3b + 2ck)x + 4a/k^2 - 2b/k + c$$

Figure 3: The set of generalized formulas for pairing quadratic roots on $kxy + x + y$

The roots of these figure 3 equations combine and reflect to zero out in $kxy + x + y = 0$. If Q's roots are real, the +/– complements pair as points on (1.3) lines as well. Yet, though Q-sets and T-matrices are no longer needed to pair roots on (1.3), in generalizations to follow they are still of good use.

Some Symmetries of $xy + bx + cy = 0$

The scope of this venture expands as rotations and reflections of $xy + x + y = 0$ open other areas of the plane. The patterns next to be considered are brought into evidence as the equation in (1.2) is generalized to:

$$xy + bx + cy = 0 \tag{1.4}$$

In the interests of symmetry and simplicity, b and c values are allowed to be either of ±1. Hence, there are four variants of (1.4) to consider:

$$xy + x + y \qquad xy - x - y \qquad xy - x + y \qquad xy + x - y$$

These equations correlate by color to the hyperbolas H_1, H_2, H_3 and H_4 in the graph below:

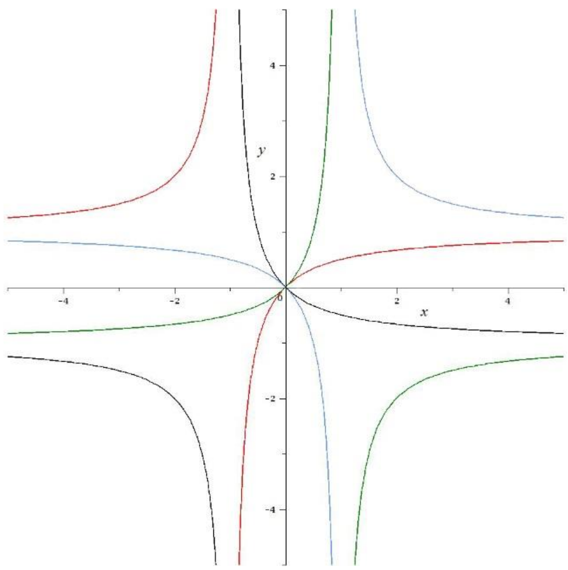

Figure 4: Reflections and rotations of $xy + x + y$

It is evident from the symmetries in figure 4 that simple signs reversals of table 2 and 3 roots will, in the three possible combinations, shift the $yx + x + y$ lines into the other three quadrants. The matrix (T_s) below is used to this end:

$$T_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.5}$$

For convenience, the matrices based on coefficients in tables 2 and 3 are labeled in order (left to right by top and bottom rows); M, M_1, M_2 and M_3. Each, in turn, is multiplied by T_S to give the set below.

$$M \cdot T_S = M_4 = \begin{pmatrix} 4 & 13 & -9 \\ 9 & 5 & -8 \\ 4 & 11 & -12 \end{pmatrix} \qquad M_1 \cdot T_S = M_5 = \begin{pmatrix} 8 & 5 & -9 \\ 6 & 11 & -8 \\ 3 & 13 & -12 \end{pmatrix}$$

$$M_2 \cdot T_S = M_6 = \begin{pmatrix} 4 & -29 & 33 \\ 9 & -41 & 38 \\ 4 & -27 & 26 \end{pmatrix} \qquad M_3 \cdot T_S = M_7 = \begin{pmatrix} 8 & -37 & 33 \\ 6 & -35 & 38 \\ 3 & -25 & 26 \end{pmatrix}$$

Table 7: T_S reflects roots on $xy + x + y$ across the line $y = -x$ to $xy - x - y$

The procedure used to find the first set of transforms is employed to find a new set:

$$M_4^{-1} \cdot M_5 = T_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \qquad M_4^{-1} \cdot M_6 = T_5 = \begin{pmatrix} 1 & -4 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_4^{-1} \cdot M_7 = T_6 = \begin{pmatrix} 1 & -4 & 4 \\ 1 & -3 & 2 \\ 1 & -2 & 1 \end{pmatrix}$$

Table 8: The transform matrix set that, with T_0, pairs quadratic roots on $xy - x - y$

These matrices too compose the Klein group. Generalizing as before:

$$T_4^k = \begin{pmatrix} 1 & 0 & 0 \\ k & -1 & 0 \\ k^2 & -2k & 1 \end{pmatrix} \qquad T_5^k = \begin{pmatrix} 1 & -4/k & 4/k^2 \\ 0 & -1 & 2/k \\ 0 & 0 & 1 \end{pmatrix} \qquad T_6^k = \begin{pmatrix} 1 & -4/k & 4/k^2 \\ k & -3 & 2/k \\ k^2 & -2k & 1 \end{pmatrix}$$

Table 9: Table 8 matrices generalized to map roots to $kxy - x - y$

Note that the net result has been to shift the negative signs in the original transforms (T_j, $j = 1..3$) from horizontal to vertical. While a similar procedure may be invoked to derive transforms for $kxy - x + y$ and $kxy + x - y$, an expedient is to apply T_S directly to the transform sets themselves. In tables 10 and 11 below, T_S applies in turn to the T_j in tables 6 and 9.

$$T_1^k \cdot T_S = T_7 = \begin{pmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ k^2 & -2k & 1 \end{pmatrix} \qquad T_2^k \cdot T_S = T_8 = \begin{pmatrix} 1 & -4/k & 4/k^2 \\ 0 & 1 & -2/k \\ 0 & 0 & 1 \end{pmatrix} \qquad T_3^k \cdot T_S = T_9 = \begin{pmatrix} 1 & -4/k & 4/k^2 \\ -k & 3 & -2/k \\ k^2 & -2k & 1 \end{pmatrix}$$

Table 10: The generalized T-matrices that map roots to $kxy - x + y$

$$T_4^k \cdot T_S = T_{10} = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k^2 & 2k & 1 \end{pmatrix} \qquad T_5^k \cdot T_S = T_{11} = \begin{pmatrix} 1 & 4/k & 4/k^2 \\ 0 & 1 & 2/k \\ 0 & 0 & 1 \end{pmatrix} \qquad T_6^k \cdot T_S = T_{12} = \begin{pmatrix} 1 & 4/k & 4/k^2 \\ k & 3 & 2/k \\ k^2 & 2k & 1 \end{pmatrix}$$

Table 11: Generalized transforms that map roots to $kxy + x - y$

These last two sets are not groups, but members of one are inverse to the other:

$$T_7^k \cdot T_{10}^k = T_8^k \cdot T_{11}^k = T_9^k \cdot T_{12}^k = 1.$$

Another transform is, for generality, included in the mix:

$$T_R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1.6}$$

T_R, the identity T_0 with signs reversed, transforms the transforms. Applied to the T-matrices it reverses the signs of each matrix and expands the four-groups to E8 (the elementary abelian group of order 8).

Note too that certain symmetries of T_1 compose elements of the dihedral/symmetric group D_3/S_3.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

Table 12: The group D_3/S_3 represented as reflections and a composition of reflections of T_1

These patterns also work with T_4 in table 8. (Are any interesting relationships to be discovered among the roots of these equations?) Table 12 entries can also be generalized using the established patterns below. For $k > 1$, D remains invariant, but the group properties are lost.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -k & -1 & 0 \\ k^2 & 2k & 1 \end{pmatrix} \quad \begin{pmatrix} k^2 & 2k & 1 \\ -k & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -k \\ 1 & 2k & k^2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2k & k^2 \\ 0 & -1 & -k \\ 0 & 0 & 1 \end{pmatrix}$$

Table 13: Elements of table 12 generalized as in table 11

The table 13 matrices, applied in order, generate the equations below.

1) $ax^2 + bx + c$	4) $ax^2 + (2a - b)x + a - b + c$
2) $cx^2 + bx + a$	5) $(a - b + c)x^2 + (2a - b)x + c$
3) $(a - b + c)x^2 + (2c - b)x + c$	6) $cx^2 + (2c - b)x + a - b + c$

Figure 5: Generalized quadratics from table 12 transforms

Let the two rightmost matrices in table 12 be designated T_{13} and T_{14}. They too are defined as ‘transforms’ in this context, because the discriminant of the equations they create (5 and 6 in figure 5) is, like the others, identical to that of equation 1. The generalizing patterns in table 13 can evidently be tacked on to almost any transform matrix, including the more general forms seen later. Interesting that for T_7 and T_{10}, these patterns correlate exactly to the exponential powers:

$$T_7^k = \begin{pmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ k^2 & -2k & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}^k \qquad T_{10}^k = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k^2 & 2k & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}^k$$

Table 14: For T_7 and T_{10}, the generalized forms match the exponential powers

A Distinguishing Property of *T*-Matrices Defined

An attribute common to (indeed, part of the definition of) T-matrices is $(a\ \ b\ \ c) \cdot T$ leaves D, the discriminant of the original equation $ax^2 + bx + c$, unchanged. To see why this is so, start with the T_j (i.e., numbered) matrices in their general forms.

$$T_D = \begin{pmatrix} s^2 & 2rs & r^2 \\ st & rt+s^2 & rs \\ t^2 & 2st & s^2 \end{pmatrix} \qquad T_E = \begin{pmatrix} r^2 & 2rs & s^2 \\ rs & rt+s^2 & st \\ s^2 & 2st & t^2 \end{pmatrix}$$
$$T_F = \begin{pmatrix} t^2 & 2st & s^2 \\ st & rt+s^2 & rs \\ s^2 & 2rs & r^2 \end{pmatrix} \qquad T_G = \begin{pmatrix} s^2 & 2st & t^2 \\ rs & rt+s^2 & st \\ r^2 & 2rs & s^2 \end{pmatrix} \tag{1.7}$$

Every T_j, $j = 1..14$, fits into the (1.7) templates, where lexical (alphabetical) order corresponds to numerical order. Note that any (1.7) matrix generates the others by vertical/horizontal flips. The chiral symmetries have effects that will be explored later. Another general consideration is signage: the purely positive pattern is seen in S; T_S operates on S to generate the set of signage patterns seen below.

$$S = \begin{pmatrix} + & + & + \\ + & + & + \\ + & + & + \end{pmatrix} \quad T_S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ST_S = \begin{pmatrix} + & + & + \\ - & - & - \\ + & + & + \end{pmatrix} \quad T_S S = \begin{pmatrix} + & - & + \\ + & - & + \\ + & - & + \end{pmatrix} \quad T_S S T_S = \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Table 15: Four T-matrix signage patterns

Multiplying by T_R completes the set:

$$ST_R = \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix} \quad ST_S T_R = \begin{pmatrix} - & - & - \\ + & + & + \\ - & - & - \end{pmatrix} \quad T_S S T_R = \begin{pmatrix} - & + & - \\ - & + & - \\ - & + & - \end{pmatrix} \quad T_S S T_S T_R = \begin{pmatrix} - & + & - \\ + & - & + \\ - & + & - \end{pmatrix}$$

Table 16: Four more patterns complete the signage set

T_R and T_S are of a form different from the T_j, but they too leave D unchanged. Next, the four matrices in (1.7) are generalized to T_H in (1.8). Lexical order is, for the purpose of the proof to follow, now unimportant; all permutations of r, s, t and u give the same result.

$$\mathbf{T_2} = \begin{pmatrix} t^2 & 2tu & u^2 \\ rt & ru+st & su \\ r^2 & 2rs & s^2 \end{pmatrix} \tag{1.8}$$

Theorem:

The quadratic equation with coefficients $(a \;\; b \;\; c) \cdot \mathbf{T_2}$ has, up to a square factor, the same discriminant as $ax^2 + bx + c$.

Proof:

Multiplying $(a \;\; b \;\; c) \cdot \mathbf{T_2}$ gives the equation

$$\left(at^2 + brt + cr^2\right)x^2 + \left(2atu + b\left(ru + st\right) + 2crs\right)x + au^2 + bsu + cs^2 \tag{1.9}$$

Solving by the quadratic formula leaves $8acrstu - 4acr^2u^2 - 4acs^2t^2 - 2b^2rstu + b^2r^2u^2 + b^2s^2t^2$ under the radical. This factors to $(b^2 - 4ac)(ru - st)^2$, leaving $b^2 - 4ac$ unchanged.

■

Of course, when $ru - st$ equals zero, the discriminant vanishes. A more general proof would also show (as trials confirm) that all of the signage patterns identified in tables 15 and 16 give this same result.

As a corollary, the proof holds for the transpose of $\mathbf{T_2}$. I.e., $(a \quad b \quad c) \cdot T_H = T_H^{\mathrm{T}} \cdot (a \quad b \quad c)^{\mathrm{T}}$ in the sense that they both produce the quadratic coefficients in (1.9).

$$\begin{pmatrix} t^2 & rt & r^2 \\ 2tu & ru+st & 2rs \\ u^2 & su & s^2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow (at^2 + brt + cr^2)x^2 + (2atu + b(ru+st) + 2crs)x + au^2 + bsu + cs^2$$

A matrix built on $r = s = t = u = j$ reduces to a singular sort of 'Pascal matrix', comprising, in this 3 x 3 case, just row 2 of the triangle. More on this type of matrix later.

$$\begin{pmatrix} j^2 & 2j^2 & j^2 \\ j^2 & 2j^2 & j^2 \\ j^2 & 2j^2 & j^2 \end{pmatrix} = j^2\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}: \;\; \begin{pmatrix} a & b & c \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} a+b+c & 2a+2b+2c & a+b+c \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$

Table 17: $r = s = t = u$ reduces quadratic coefficients to those of the expansion of $(x + 1)^2$

T-Matrices as Sequence Generators

Next to be examined are sequences that emerge when $\mathbf{T_2}$-type matrices are taken to successive powers. A $\mathbf{T_2}$ matrix property is that it has, under exponentiation, an invariant form. As an example, the matrices below show that corner values remain squares as $\mathbf{T_2}$ is squared and cubed. This establishes the pattern, and it will be from the corners that sequence terms are taken.

$$\mathbf{T}_2^2 = \begin{pmatrix} (ru+t^2)^2 & \bullet & u^2(s+t)^2 \\ \bullet & \bullet & \bullet \\ r^2(s+t)^2 & \bullet & (ru+s^2)^2 \end{pmatrix}$$

$$\mathbf{T}_2^3 = \begin{pmatrix} (2rtu+rsu+t^3)^2 & \bullet & u^2(ru+s^2+st+t^2)^2 \\ \bullet & \bullet & \bullet \\ r^2(ru+s^2+st+t^2)^2 & \bullet & (2rsu+rtu+s^3)^2 \end{pmatrix} \tag{1.10}$$

Now for some numerical examples. T_{10}, with its uniformly positive signage, is first to be explored. Flipped on a horizontal axis, it creates a matrix (T_{15}), the powers of which are representable as squares, products and sums of successive Fibonacci (φ) numbers ([A000045](A000045)). Corners are read in the order $T_{3,3}$, $T_{3,1}/T_{1,3}$ and $T_{1,1}$. Let F_n represent the nth term of the φ-sequence, and note that T_{15} in table 18 is the same as T_F in (1.7) with $r = F_0$, $s = F_1$ and $t = F_2$.

$$T_{15} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad T_{15}^2 = \begin{pmatrix} 4 & 4 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix} \quad T_{15}^3 = \begin{pmatrix} 9 & 12 & 4 \\ 6 & 7 & 2 \\ 4 & 4 & 1 \end{pmatrix} \quad T_{15}^4 = \begin{pmatrix} 25 & 30 & 9 \\ 15 & 19 & 6 \\ 9 & 12 & 4 \end{pmatrix} \cdots$$

Table 18: T_{15} to successive powers generates the squares of successive φ-sequence terms

In table 18, $F_0 = 0$ as the initial of the three consecutive F_n results in just one new squared term for each increment of the exponent. Starting with T_{15}^3 taken to successive powers gives the squares of three such terms for every increment.

$$(T_{15}^3)^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_{15}^3 = \begin{pmatrix} 9 & 12 & 4 \\ 6 & 7 & 2 \\ 4 & 4 & 1 \end{pmatrix} \quad (T_{15}^3)^2 = \begin{pmatrix} 169 & 208 & 64 \\ 104 & 129 & 40 \\ 64 & 80 & 25 \end{pmatrix} \quad (T_{15}^3)^3 = \begin{pmatrix} 3025 & 3740 & 1156 \\ 1870 & 2311 & 714 \\ 1156 & 1428 & 441 \end{pmatrix}$$

Table 19: T_{15}^3 to successive powers generates the squares of the next three φ-sequence terms

Taking T_{15}^3 to negative powers has the effect of imposing the inverse signage pattern and reversing the order of the numbers in the corners on the main diagonals of the squares.

$$(T_{15}^3)^{-1} = \begin{pmatrix} 1 & -4 & 4 \\ -2 & 7 & -6 \\ 4 & -12 & 9 \end{pmatrix} \quad (T_{15}^3)^{-2} = \begin{pmatrix} 25 & -80 & 64 \\ -40 & 129 & -104 \\ 64 & -208 & 169 \end{pmatrix} \quad (T_{15}^3)^{-3} = \begin{pmatrix} 441 & -1428 & 1156 \\ -714 & 2311 & -1870 \\ 1156 & -3740 & 3025 \end{pmatrix}$$

Table 20: The effect of negative exponents on T_{15}^3

The change in corner values relates to other changes, which correspond to flipping the matrix on both the horizontal and vertical axes. Reading in the same fashion then gives numbers to the left of zero in reverse order to those on the right.

The terms of the φ-sequence to the left of zero have alternating signs. A question for future reference is; for which of the squares in the corners do we take the negative roots? The truncated Fibonacci sequence –8, 5, –3, 2, –1, 1, 0 corresponds to F_{-6}, F_{-5}, F_{-4}, F_{-3}, F_{-2}, F_{-1}, F_0; thus an even negative index means the term has a negative sign. Now, referencing the matrices in table 20, we see that an odd exponent accords with negative roots in $T_{3,3}$ and $T_{1,1}$, and positive in $T_{3,1}/T_{1,3}$. An even exponent means positive roots in $T_{3,3}$ and $T_{1,1}$, and negative in $T_{3,1}/T_{1,3}$. We'll return to this later…

A generalized form of this matrix (T_φ) is in (1.11) below.

$$T_\varphi^n = \begin{pmatrix} F_{n+1}^2 & 2F_nF_{n+1} & F_n^2 \\ F_nF_{n+1} & F_nF_{n+1}+F_{n-1}^2 & F_nF_{n-1} \\ F_n^2 & 2F_nF_{n-1} & F_{n-1}^2 \end{pmatrix} \tag{1.11}$$

T_{10} is now squared, and again flipped on a horizontal axis.

$$T_{16} = \begin{pmatrix} 4 & 4 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad T_{16}^2 = \begin{pmatrix} 25 & 20 & 4 \\ 10 & 9 & 2 \\ 4 & 4 & 1 \end{pmatrix} \quad T_{16}^3 = \begin{pmatrix} 144 & 120 & 25 \\ 60 & 49 & 10 \\ 25 & 20 & 4 \end{pmatrix} \quad T_{16}^4 = \begin{pmatrix} 841 & 696 & 144 \\ 348 & 289 & 60 \\ 144 & 120 & 25 \end{pmatrix} \quad \cdots$$

Table 21: T_{16} to successive powers generates the squares of successive Pell numbers

This is the basis for a matrix that does for Pell numbers (0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741… A000129) what T_{15}^n does for Fibonacci numbers.

$$P_\varphi^n = \begin{pmatrix} P_{n+1}^2 & 2P_nP_{n+1} & P_n^2 \\ P_nP_{n+1} & P_nP_{n+1}+P_{n-1}^2 & P_nP_{n-1} \\ P_n^2 & 2P_nP_{n-1} & P_{n-1}^2 \end{pmatrix} \tag{1.12}$$

T_{16}^{3n} gives three Pell numbers for every increment of n.

$$T_{16}^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_{16}^3 = \begin{pmatrix} 144 & 120 & 25 \\ 60 & 49 & 10 \\ 25 & 20 & 4 \end{pmatrix} \quad T_{16}^6 = \begin{pmatrix} 28561 & 23660 & 4900 \\ 11830 & 9801 & 2030 \\ 4900 & 4060 & 841 \end{pmatrix} \quad T_{16}^6 = \begin{pmatrix} 5654884 & 4684660 & 970225 \\ 2342330 & 1940449 & 401880 \\ 970225 & 803760 & 166464 \end{pmatrix}$$

Table 22: T_{16}^{3n} generates three terms of the Pell sequence for each increment of n

T_{16} to negative powers gives the now-familiar patterns below.

$$T_{16}^{-3} = \begin{pmatrix} 4 & -20 & 25 \\ -10 & 49 & -60 \\ 25 & -120 & 144 \end{pmatrix} \quad T_{15}^{-6} = \begin{pmatrix} 841 & -4060 & 4900 \\ -2030 & 9801 & -11830 \\ 4900 & -23660 & 28561 \end{pmatrix} \quad T_{15}^{-9} = \begin{pmatrix} 166464 & -803760 & 970225 \\ -401880 & 1940449 & -2342330 \\ 970225 & -4684660 & 5654884 \end{pmatrix}$$

Table 23: T_{16}^{-3n} gives the familiar rotational symmetry and signage pattern

	P_9	P_8	P_7	P_6	P_5	P_4	P_3	P_2	P_1	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	
…	985	–408	169	–70	29	–12	5	–2	1	0	1	2	5	12	29	70	169	408	985	…

Table 24: A few terms of the Pell sequence

Table 24 shows that again an even negative index means the term has a negative sign. And, as before, table 22 shows that an odd exponent accords with negative square roots in $T_{3,3}$ and $T_{1,1}$, and positive in $T_{3,1}/T_{1,3}$. An even exponent means positive roots in $T_{3,3}$ and $T_{1,1}$, and negative in $T_{3,1}/T_{1,3}$.

Generalizing, T_{10}^{k} flipped on a horizontal axis generates $F_{n-1} + kF_{n-1} = F_{n+1}$ (e.g., $k = 3$ gives A006190)..

Next, vertical axis flips are used to explore the T_F/T_G chirality seen in (1.7). The T_F-type matrix T_{15}^{2} in table 17 is reflected vertically to the T_G form to create T_{17} in table 25.

$$T_{17}^{0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_{17} = \begin{pmatrix} 1 & 4 & 4 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix} \quad T_{17}^{2} = \begin{pmatrix} 9 & 24 & 16 \\ 6 & 17 & 12 \\ 4 & 12 & 9 \end{pmatrix} \quad T_{17}^{3} = \begin{pmatrix} 49 & 140 & 100 \\ 35 & 99 & 70 \\ 25 & 70 & 49 \end{pmatrix} \quad T_{17}^{4} = \begin{pmatrix} 289 & 816 & 576 \\ 204 & 577 & 408 \\ 144 & 408 & 289 \end{pmatrix}$$

Table 25: T_{17}, chiral to T_{15}^{2}, generates a distinctive kind of sequence

The corners of these matrices, now read in the order $T_{3,1}$, $T_{3,3}/T_{1,1}$ and $T_{1,3}$, are squares of the sequence A249576: 0, 1, 0, 1, 1, 2, 2, 3, 4, 5, 7, 10, 12, 17, 24…

T_{17}, taken to negative powers, gives the matrices below.

$$T_{17}^{-1} = \begin{pmatrix} 1 & -4 & 4 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{pmatrix} \quad T_{17}^{-2} = \begin{pmatrix} 9 & -24 & 16 \\ -6 & 17 & -12 \\ 4 & -12 & 9 \end{pmatrix} \quad T_{17}^{-3} = \begin{pmatrix} 49 & -140 & 100 \\ -35 & 99 & -70 \\ 25 & -70 & 49 \end{pmatrix} \quad T_{17}^{-4} = \begin{pmatrix} 289 & -816 & 576 \\ -204 & 577 & -408 \\ 144 & -408 & 289 \end{pmatrix}$$

Table 26: T_{17} to successive negative powers

Taking the table 19 Fibonacci matrices to negative powers in table 20 had (beyond the signage changes) the effect of 'transposing' T_{15}^{3n} across the minor diagonal. Thus, when table 20 matrices are read in the same order as table 19 entries, terms to the left of zero are in reverse order to those on the right. However, taking table 25 entries to negative powers has a different effect, and when table 26 matrices are read in the same direction, there's a marked difference in the order to the left of zero. The sequence below is A249576 extended to the left. The significance of the three alternating colors is noted anon.

…–70, 99, –140, 29, –41, 58, –12, 17, –24, 5, –7, 10, –2, 3, –4, 1, –1, 2, 0, 1, 0, 1, 1, 2, 2, 3, 4, 5, 7, 10, 12, 17, 24, 29, 41, 58, 70, 99, 140, 169, 239, 338, 408, 577, 816, 985, 1393, 1970, 2378, 3363, 4756, 5741, 8119, 11482, 13860, 19601, 27720, 33461, 47321, 66922, 80782, 114243, 161564, 195025, 275807…

Left-of-zero numbers, aligned in 'packets' of three, ascend in absolute value from left to right. While this pattern is unusual, note that it 'factors' into three sequences of a more conventional form.

–70, 29, –12, 5, –2, 1, 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025…

99, –41, 17, –7, 3, –1, 1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119, 19601, 47321, 114243, 275807…

–140, 58, –24, 10,–4, 2, 0, 2, 4, 10, 24, 58, 140, 338, 816, 1970, 4756, 11482, 27720, 66922, 161564…

Sequences within sequences… defining a_n as the nth term of A249576, note that:

a_{3n} = 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025… = A000129

a_{3n+1} = 1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119, 19601, 47321, 114243, 275807… = A001333

a_{3n+2} = 0, 2, 4, 10, 24, 58, 140, 338, 816, 1970, 4756, 11482, 27720, 66922, 161564… = A163271

Taking the negatives exponents to higher values and reversing the order: A249577 = 2, -1, 1, -4, 3, -2, 10, -7, 5, -24, 17, -12, 58, -41, 29, -140, 99, -70, 338, -239, 169, -816, 577, -408, 1970, -1393, 985, -4756…

A lot of sequences for the effort of one. To continue in this fashion, T_{15}^{3} in table 17 reflects vertically to create T_{18} in table 26.

$$T_{18} = \begin{pmatrix} 4 & 12 & 9 \\ 2 & 7 & 6 \\ 1 & 4 & 4 \end{pmatrix} \quad T_{18}^{2} = \begin{pmatrix} 49 & 168 & 144 \\ 28 & 97 & 84 \\ 16 & 56 & 49 \end{pmatrix} \quad T_{18}^{3} = \begin{pmatrix} 676 & 2340 & 2025 \\ 390 & 1351 & 1170 \\ 225 & 780 & 676 \end{pmatrix} \quad T_{18}^{4} = \begin{pmatrix} 9409 & 32592 & 28224 \\ 5432 & 18817 & 16296 \\ 3136 & 10864 & 9409 \end{pmatrix}$$

Table 27: T_{18}, chiral to T_{15}^{3}, generates another compound sequence

0, 1, 0, 1, 2, 3, 4, 7, 12, 15, 26, 45, 56, 97, 168, 209, 362, 627, 780, 1351, 2340, 2911, 5042… = A249578.

Again, the main sequence factors to three others. I.e., a_{3n} = A001353; a_{3n+1} = A001075; a_{3n+2} = A005320.

To move on, we reference $\mathbf{T_2}$ in (1.8). So far, the base matrices have derived from Fibonacci-type T-matrices and were limited to three variables. But $\mathbf{T_2}$ accepts an arbitrary number in each corner. First is just to break the $T_{3,3} = T_{1,1}$ symmetry with $r = 1$, $s = 1$, $t = 2$, $u = 3$.

$$T_{19} = \begin{pmatrix} 4 & 12 & 9 \\ 2 & 5 & 3 \\ 1 & 2 & 1 \end{pmatrix} \quad T_{19}^{2} = \begin{pmatrix} 49 & 126 & 81 \\ 21 & 55 & 36 \\ 9 & 24 & 16 \end{pmatrix} \quad T_{19}^{3} = \begin{pmatrix} 529 & 1380 & 900 \\ 230 & 599 & 390 \\ 100 & 260 & 169 \end{pmatrix} \quad T_{19}^{4} = \begin{pmatrix} 5776 & 15048 & 9801 \\ 2508 & 6535 & 4257 \\ 1089 & 2838 & 1849 \end{pmatrix}$$

Table 28: T_{19} breaks the usual $T_{3,3} = T_{1,1}$ symmetry

The table 27 matrices (and the identity) generate the sequence 0, 1, 1, 0, 1, 1, 2, 3, 3, 4, 7, 9, 10, 13, 23, 30, 33, 43, 76, 99, 109, 142, 251, 327, 360, 469, 829, 1080, 1189, 1549, 2738, 3567, 3927… = A249579.

Negative exponents generate entries in table 29.

$$T_{19}^{-1}=\begin{pmatrix}1 & -6 & 9\\ -1 & 5 & -6\\ 1 & -4 & 4\end{pmatrix}\quad T_{19}^{-2}=\begin{pmatrix}16 & -72 & 81\\ -12 & 55 & -63\\ 9 & -42 & 49\end{pmatrix}\quad T_{19}^{-3}=\begin{pmatrix}169 & -780 & 900\\ -130 & 599 & -690\\ 100 & -460 & 529\end{pmatrix}\quad T_{19}^{-4}=\begin{pmatrix}1849 & -8514 & 9801\\ -1419 & 6535 & -7524\\ 1089 & -5016 & 5776\end{pmatrix}$$

Table 29: T_{19} to negative powers

Following the protocols for taking negative roots from alternate corners in alternate matrices that were derived earlier from table 20, we get this signage: …–33, 76, 43, –99, 10, –23, –13, 30, –3, 7, 4, –9, 1, –2, –1, 3… (These numbers in reverse order are A249580.) The pattern, of period 8, reads from right to left:

$$-++-+--+ \tag{1.13}$$

This pattern, unusual and perhaps unique, is a glimpse of a bigger picture. Note that all four factor sequences have the usual alternating signs. Yet only those with a zero are numerical mirror images; the other two, green and blue, are 'cross-wired'.

…–360, 109, –33, 10, –3, 1, 0, 1, 3, 10, 33, 109, 360, 1189… purely positive terms ($a_{n>0}$) are in A006190

…829, –251, 76, –23, 7, –2, 1, 1, 4, 13, 43, 142, 469, 1549… $a_{n>0}$ in A003688

…469, –143, 43, –13, 4, –1, 1, 2, 7, 23, 76, 251, 829, 2738… $a_{n>0}$ in A052924

…–1080, 327, –99, 30, –9, 3, 0, 3, 9, 30, 99, 327, 1080, 3567… $a_{n>0}$ in A052906 = 3·A006190

Moreover, removing one of the red duplicates from – + + – + – – + – + + – + – – + again leaves alternating signs: – + – + – + – + – + – +. This is what happens, e.g., in matrices where $T_{1,3} = T_{3,1}$ and only one value is recorded.

Another confirmation of this pattern's accuracy is that the identity $a_n = 3a_{n\text{-}4} + a_{n\text{-}8}$ (supplied for A249579 by Colin Barker) works everywhere in the sequence. The color-coded versions makes this easy to verify…

…–360, 829, 469, –1080, 109, –251, –142, 327, –33, 76, 43, –99, 10, –23, –13, 30, –3, 7, 4, –9, 1, –2, –1, 3, 0, 1, 1, 0, 1, 1, 2, 3, 3, 4, 7, 9, 10, 13, 23, 30, 33, 43, 76, 99, 109, 142, 251, 327, 360, 469, 829, 1080, 1189, 1549, 2738, 3567…

Moreover, A100638 utilizes a 2 x 2 matrix that generates same sequences as the 3 x 3 in (1.8). This smaller matrix returns not the squares of sequence terms, but the terms themselves.

$$\mathbf{T}_1=\begin{pmatrix}t & u\\ r & s\end{pmatrix} \tag{1.14}$$

E.g., let r, s, t, u = 1,1,2,3 and take (1.14) to successive powers:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 7 & 9 \\ 3 & 4 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} 23 & 30 \\ 10 & 13 \end{pmatrix}, \qquad A^4 = \begin{pmatrix} 33 & 43 \\ 76 & 99 \end{pmatrix} \cdots$$

This is A249579 again. As A is taken to successive negative powers, any signage ambiguities are resolved.

$$A^{-1} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}, \qquad A^{-2} = \begin{pmatrix} 4 & -9 \\ -3 & 7 \end{pmatrix}, \qquad A^{-3} = \begin{pmatrix} -13 & 30 \\ 10 & -23 \end{pmatrix} \qquad A^{-4} = \begin{pmatrix} 43 & -99 \\ -33 & 76 \end{pmatrix} \cdots$$

Sequence Generation Proceeds Using T_1-Type Matrices

Utilizing the simpler 2 x 2 matrix form in (1.14), something of note becomes apparent as A above is flipped on a vertical axis to create T_{20}.

$$T_{20} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \qquad T_{20}^2 = \begin{pmatrix} 11 & 8 \\ 4 & 3 \end{pmatrix}, \qquad T_{20}^3 = \begin{pmatrix} 41 & 30 \\ 15 & 11 \end{pmatrix}, \qquad T_{20}^4 = \begin{pmatrix} 153 & 112 \\ 56 & 41 \end{pmatrix} \cdots$$

$$T_{20}^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}, \qquad T_{20}^{-2} = \begin{pmatrix} 3 & -8 \\ -4 & 11 \end{pmatrix}, \qquad T_{20}^{-3} = \begin{pmatrix} 11 & -30 \\ -15 & 41 \end{pmatrix}, \qquad T_{20}^{-4} = \begin{pmatrix} 41 & -112 \\ -56 & 153 \end{pmatrix} \cdots$$

Note that minus signs no longer alternate, but now lie on the antidiagonal. The sequence this creates is:

…571, –209, –418, 153, 153, –56, –112, 41, 41, –15, –30, 11, 11, –4, –8, 3, 3, –1, –2, 1, 1, 0, 0, 1, 1, 1, 2, 3, 3, 4, 8, 11, 11, 15, 30, 41, 41, 56, 112, 153, 153, 209, 418, 571…

The signage pattern now, reading from right to left, is of period 4:

$$- + + - \tag{1.15}$$

A closer look at two (1.14) inverses explains this pattern.

$$\mathbf{T}_1^{-1} = \begin{pmatrix} -\dfrac{s}{ru-st} & \dfrac{u}{ru-st} \\ \dfrac{r}{ru-st} & -\dfrac{t}{ru-st} \end{pmatrix}, \qquad \mathbf{T}_1^{-2} = \begin{pmatrix} \dfrac{ru+s^2}{r^2u^2-2rstu+s^2t^2} & -\dfrac{u(s+t)}{r^2u^2-2rstu+s^2t^2} \\ -\dfrac{r(s+t)}{r^2u^2-2rstu+s^2t^2} & \dfrac{ru+t^2}{r^2u^2-2rstu+s^2t^2} \end{pmatrix} \tag{1.16}$$

If in the first (T^{-1}) case, $|ru| < |st|$ then [1,2] and [2,1] are negative and [1,1] and [2,2] are positive. Then in the T^{-2} case the denominator is positive so [1,2] and [2,1] are negative and [1,1] and [2,2] positive again.

The color scheme makes it easy to see that $a_n = 4a_{n-4} - a_{n-8}$.

…571, –209, –418, 153, 153, –56, –112, 41, 41, –15, –30, 11, 11, –4, –8, 3, 3, –1, –2, 1, 1, 0, 0, 1, 1, 1, 2, 3, 3, 4, 8, 11, 11, 15, 30, 41, 41, 56, 112, 153, 153, 209, 418, 571…

Interesting properties of this sequence become more evident as it is factorized to its four constituents, f_j, $j = 1..4$. Note first that $f_1 = f_4$, $f_3 = 2f_3$ and for all f_j, $|a_{-n}| = |a_n|$. However, per (1.16), the +/– signs are no longer evenly distributed: i.e., f_1 and f_4 are uniformly positive, but f_2 and f_3 are positive to the right of zero and negative to the left. In the earliest sequences (e.g., A249576), one of the adjacent duplicate terms was omitted, and the period 2 (alternating) signage pattern was seen. But omitting a duplicate here gives the period 3 pattern (from right to left) + – –.

The formula now is $a_n = 4a_{n\text{-}1} - a_{n\text{-}2}$.

f_1 = …571, 153, 41, 11, 3, 1, 1, 3, 11, 41, 153…

f_2 = …–209, –56, –15, –4, –1, 0, 1, 4, 15, 56, 209… $a_{n>0}$ in A001353

f_3 = …–418, –112, –30, –8, –2, 0, 2, 8, 30, 112, 418…= 2 · A001353

f_4 = …153, 41, 11, 3, 1, 1, 3, 11, 41, 153, 571…

The matrices below show, for two incrementations, how the relationship between the terms in [1,2] and [2,1] (i.e., r and u) remains constant under exponentiation.

$$\mathbf{T_1} = \begin{pmatrix} \bullet & u \\ r & \bullet \end{pmatrix}, \quad \mathbf{T_1^2} = \begin{pmatrix} \bullet & u(s+t) \\ r(s+t) & \bullet \end{pmatrix}, \quad \mathbf{T_1^3} = \begin{pmatrix} \bullet & u(ru+s^2+st+t^2) \\ r(ru+s^2+st+t^2) & \bullet \end{pmatrix} \quad (1.17)$$

This investigation continues with $r = 1$, $s = 2$, $t = 3$, $u = 4$.

$$T_{21} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \quad T_{21}^2 = \begin{pmatrix} 13 & 20 \\ 5 & 8 \end{pmatrix} \quad T_{21}^3 = \begin{pmatrix} 59 & 92 \\ 23 & 36 \end{pmatrix} \quad T_{21}^4 = \begin{pmatrix} 269 & 420 \\ 105 & 164 \end{pmatrix} \cdots$$

Table 30: T_{21} is the first example with all corner values distinct

The cyclical colors simplify confirming the identity $a_n = 5a_{n\text{-}4} - 2a_{n\text{-}8}$ (from Colin Barker, A249581).

0, 1, 1, 0, 1, 2, 3, 4, 5, 8, 13, 20, 23, 36, 59, 92, 105, 164, 269, 420, 479, 748, 1227, 1916, 2185, 3412, 5597, 8740, 9967, 15564, 25531, 39868, 45465, 70996, 116461, 181860, 207391, 323852, 531243, 829564, 946025, 1477268, 2423293, 3784100…

Factored:

0, 1, 5, 23, 105, 479, 2185, 9967, 45465, 207391, 946025… A107839

1, 2, 8, 36, 164, 748, 3412, 15564, 70996, 323852, 1477268… A147722

1, 3, 13, 59, 269, 1227, 5597, 25531, 116461, 531243, 2423293… A052984

0, 4, 20, 92, 420, 1916, 8740, 39868, 181860, 829564, 3784100… = 4 · A107839

For this selection of r, s, t and u values, terms to the left of zero are most often fractional, but certain choices of r, s, t, u will always give integers for negative exponents.

To see why, take the inverse of $\mathbf{T_2}$ in (1.8). the entries all have $(ru - st)^2$ as a denominator.

If $r, s, t, u = j–1, j, j, j+1$, then $((j–1)(j+1) – j^2)^2 = 1$.

If $r, s, t, u = F_{n–1}, F_n, F_n, F_{n+1}$, then the identity $F_n^2 – F_{n–1} \cdot F_{n+1} = (–1)^{n+1}$ gives unity again.

If r, s, t, u are consecutive F_n, $(ru – st)^2$ is the left side of the identity $F_n \cdot F_{n+3} – F_{n+1} \cdot F_{n+2} = (–1)^{n+1}$.

Of course, there are also countless other number combinations that solve $(ru – st)^2 = 1$; e.g., $(4 \cdot 5 – 3 \cdot 7)$ and $(2 \cdot 17 – 5 \cdot 7)$ are two.

To continue with a few more examples, let $r, s, t, u = 1, 2, 3, 5$.

$$T_{22} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

…–12649, 35307, 22658, –63245, –2640, 7369, 4729, –13200, –551, 1538, 987, –2755, –115, 321, 206, –575, –24, 67, 43, –120, –5, 14, 9, –25, –1, 3, 2, –5, 0, 1, 1, 0, 1, 2, 3, 5, 5, 9, 14, 25, 24, 43, 67, 120, 115, 206, 321, 575, 551, 987, 1538, 2755, 2640, 4729, 7369, 13200, 12649, 22658, 35307, 63245…

A formula for this sequence is $a_n = 5a_{n-4} – a_{n-8}$. For the constituents below, $a_n = 5a_{n-1} – a_{n-2}$. The period 4 signage pattern above gives the following factors.

…–12649, –2640, –551, –115, –24, –5, –1, 0, 1, 5, 24, 115, 551, 2640, 12649… $a_{n>0}$ in A004254

35307, 7369, 1538, 321, 67, 14, 3, 1, 2, 9, 43, 206, 987, 4729, 22658… $a_{n>0}$ in A002310

22658, 4729, 987, 206, 43, 9, 2, 1, 3, 14, 67, 321, 1538, 7369, 35307… $a_{n>0}$ in A002320

–63245, –13200, –2755, –575, –120, –25, –5, 0, 5, 25, 120, 575, 2755, 13200, 63245… = 5 · A004254

T_{22}, flipped horizontally, is T_{23}.

$$T_{23} = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$$

These same Fibonacci numbers again give integers on the left side of the zeros. The period 8 signage gives the familiar period 2 in the factor sequences. The formula $6a_{4n} + a_{4(n–1)} = 6a_{4(n+1)}$ works throughout.

…1697233, –657552, –986328, 382129, –275423, 106706, 160059, –62011, 44695, –17316, –25974, 10063, –7253, 2810, 4215, –1633, 1177, –456, –684, 265, –191, 74, 111, –43, 31, –12, –18, 7, –5, 2, 3, –1, 1, 0, 0, 1, 1, 2, 3, 5, 7, 12, 18, 31, 43, 74, 111, 191, 265, 456, 684, 1177, 1633, 2810, 4215, 7253, 10063, 17316, 25974, 44695, 62011, 106706, 160059, 275423, 382129, 657552, 986328, 1697233, 2354785, 4052018, 6078027, 10458821, 14510839, 24969660, 37454490, 64450159…

…1697233, –275423, 44695, –7253, 1177, –191, 31, –5, 1, 1, 7, 43, 265, 1633, 10063, 62011, 382129, 2354785, 14510839… $a_{n>0}$ in A015451

…–657552, 106706, –17316, 2810, –456, 74, –12, 2, 0, 2, 12, 74, 456, 2810, 17316, 106706, 657552, 4052018, 24969660… $a_{n>0}$ in A078469; to divide this sequence by 2 gives A005668

…–986328, 160059, –25974, 4215, –684, 111, –18, 3, 0, 3, 18, 111, 684, 4215, 25974, 160059, 986328, 6078027, 37454490… to divide this sequence by 3 gives A005668

…382129, –62011, 10063, –1633, 265, –43, 7, –1, 1, 5, 31, 191, 1177, 7253, 44695, 275423, 1697233, 10458821, 64450159… by absolute value, this is A015451 in reverse

The T_1/T_2 Relationship Implies a Family of Matrices

The relationship between these 2 x 2 and 3 x 3 matrices implies the existence of a larger, possibly infinite, family of *n* x *n* matrices. Since $\mathbf{T}_1^n$ generates four terms of a sequence (S) and $\mathbf{T}_2^n$ has the squares of those same four terms in its corners (call this sequence S^2), perhaps there is, for any integer n, an $(n+1)^2$ matrix ($\mathbf{T}_n$) with corners containing terms of S^n, i.e., the sequence comprising the nth power of $\mathbf{T}_1$ terms. We'll add one more matrix to those already extant, to help in the derivation of three more.

$$\mathbf{T}_0 = (1) \qquad \mathbf{T}_1 = \begin{pmatrix} t & u \\ r & s \end{pmatrix} \qquad \mathbf{T}_2 = \begin{pmatrix} t^2 & 2tu & u^2 \\ rt & ru+st & su \\ r^2 & 2rs & s^2 \end{pmatrix} \tag{1.18}$$

Let $r = s = t = u = 1$ in the matrices above, and we get the trio in table 30 below.

$$\mathbf{T}_0 = (1) \qquad \mathbf{T}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{T}_2 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Table 31: The matrices in (1.18) with $r = s = t = u = 1$

Although it may seem, at the moment, a bit of a long shot, note that the rows of a $\mathbf{T}_n$ matrix in table 31 correspond to the nth row of Pascal's triangle (P).

0										1									
1									1		1								
2								1		2		1							
3							1		3		3		1						
4						1		4		6		4		1					
5					1		5		10		10		5		1				
6				1		6		15		20		15		6		1			
7			1		7		21		35		35		21		7		1		
8		…		…		…		…		…		…		…		…		…	

Table 32: The first few rows of Pascal's triangle (P)

This gives us a start on deriving a 4 x 4 matrix that has a cubic term in each corner and thus generates the sequence S^3. Note that dots in the following matrices represent undefined elements.

$$i) = \begin{pmatrix} t^3 & \bullet & \bullet & u^3 \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ r^3 & \bullet & \bullet & s^3 \end{pmatrix} \qquad ii) = \begin{pmatrix} t^3 & \bullet & \bullet & u^3 \\ rt^2 & \bullet & \bullet & su^2 \\ r^2t & \bullet & \bullet & s^2u \\ r^3 & \bullet & \bullet & s^3 \end{pmatrix} \qquad iii) = \begin{pmatrix} t^3 & 3t^2u & 3tu^2 & u^3 \\ rt^2 & \bullet & \bullet & su^2 \\ r^2t & \bullet & \bullet & s^2u \\ r^3 & 3r^2s & 3rs^2 & s^3 \end{pmatrix}$$

Table 33: Filling in the blanks of the 4 x 4

In *ii* a pattern from $\mathbf{T_2}$ is extrapolated to fill in the sides. In *iii*, the third row of Pascal's triangle (1,3,3,1) provides the scalars.

Now, to test out this tentative pattern, set r, s, t, u = 1, 2, 3, 4 and square the matrix.

$$\begin{pmatrix} 27 & 108 & 144 & 64 \\ 9 & 0 & 0 & 32 \\ 3 & 0 & 0 & 16 \\ 1 & 6 & 12 & 8 \end{pmatrix}^2 = \begin{pmatrix} 2197 & 3300 & 4656 & 8000 \\ 275 & 1164 & 1680 & 832 \\ 97 & 420 & 624 & 320 \\ 125 & 156 & 240 & 512 \end{pmatrix}$$

The cube roots of the corner values are 5, 8, 13 and 20, the result for those same r, s, t, u when squaring $\mathbf{T_1}$. However, a closer look shows that the edge patterns no longer conform to the *ii* form. Also, as this test matrix is taken to powers > 2, the corners no longer have integer cube roots. So the next step is to find general expressions for the interior terms.

For this purpose, *iii* is fitted with e, f, g and h in [2,2], [2,3], [3,2] and [3,3] as symbols to be solved for. This matrix squared is in table 34 below.

$$\begin{pmatrix} (ru+t^2)^3 & 3r^2su^3+3t^5u+3et^2u+3gtu^2 & 3rs^2u^3+3t^4u^2+3ft^2u+3htu^2 & u^3(s+t)^3 \\ r^3su^2+rt^5+ert^2+fr^2t & 3r^2s^2u^2+3rt^4u+e^2+fg & 3rs^3u^2+3rt^3u^2+ef+fh & rt^2u^3+s^4u^2+esu^2+fs^2u \\ r^3s^2u+r^2t^4+grt^2+hr^2t & 3r^2s^3u+3r^2t^3u+eg+gh & 3r^2t^2u^2+3rs^4u+fg+h^2 & r^2tu^3+s^5u+gsu^2+hs^2u \\ r^3(s+t)^3 & 3r^3t^2u+3r^2s^4+3er^2s+3grs^2 & 3r^3tu^2+3rs^5+3fr^2s+3hrs^2 & (ru+s^2)^3 \end{pmatrix}$$

Table 34: *iii*, with e, f, g and h in its interior, squared

Note that [2,1] and [2,4] both contain just the two unknowns, e and f. Referencing the *iii* pattern in table 33, [2,1] = $(ru + t^2)^2 \cdot r(s + t)$ and [2,4] = $u^2(s + t)^2 \cdot (ru + s^2)$. This give the simultaneous equations:

$$r^3su^2 + rt^5 + ert^2 + fr^2t = \left(ru + t^2\right)^2 \cdot r(s+t) \tag{1.19}$$

$$rt^2u^3 + s^4u^2 + esu^2 + fs^2u = u^2\left(s+t\right)^2 \cdot \left(ru + s^2\right) \tag{1.20}$$

To solve, say, (1.19) for e in terms of f, then substituting into (1.20) gives $f = u(ru + 2st)$. Plug that back into (1.19) for $e = t(2ru + st)$. Then [3,1] and [3,4] are likewise manipulated to find the entries for c and d. This gives $\mathbf{T_3}$ in (1.21) below.

$$\mathbf{T}_3 = \begin{pmatrix} t^3 & 3t^2u & 3tu^2 & u^3 \\ rt^2 & t(2ru+st) & u(ru+2st) & su^2 \\ r^2t & r(ru+2st) & s(2ru+st) & s^2u \\ r^3 & 3r^2s & 3rs^2 & s^3 \end{pmatrix} \tag{1.21}$$

$\mathbf{T}_3$ is the fourth member of this young family. Which of the $\mathbf{T}_2$ properties does it inherit? For one, the proof on page 10 shows that $\mathbf{T}_2$, operating on a quadratic equation's coefficients, doesn't change the core value of the discriminant $D_2 = (b^2 - 4ac)$, but increases it by a square factor. I.e., $D_2 \cdot (ru - st)^2$. For a cubic equation with coefficients a, b, c and d, the discriminant is

$$D_3 = 18abcd - 27a^2d^2 - 4ac^3 - 4b^3d + b^2c^2 \tag{1.22}$$

Multiply the cubic polynomial's coefficient matrix $(a\ b\ c\ d)$ by $\mathbf{T}_3$ and put the new coefficients into the general cubic formula: the discriminant now factors to $D_3 \cdot (ru - st)^6$.

Recall that $\mathbf{T}_2$ was discovered in the course of finding methods for pairing quadratic roots on hyperbolic curves. Are there $\mathbf{T}_3$ analogs that match cubic roots on 3D surfaces or curves? Do any nontrival forms of $\mathbf{T}_3$, numerical or symbolic, form a finite group?

Another question is, how many signage patterns can $\mathbf{T}_3$ assume and still retain its distinctive properties? Inverses of nonsingular numerical forms give two examples.

$$\begin{pmatrix} 27 & 135 & 225 & 125 \\ 9 & 48 & 85 & 50 \\ 3 & 17 & 32 & 20 \\ 1 & 6 & 12 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} 8 & -60 & 150 & -125 \\ -4 & 32 & -85 & 75 \\ 2 & -17 & 48 & -45 \\ -1 & 9 & -27 & 27 \end{pmatrix}, \quad \begin{pmatrix} 125 & 225 & 135 & 27 \\ 50 & 85 & 48 & 9 \\ 20 & 32 & 17 & 3 \\ 8 & 12 & 6 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 9 & -27 & 27 \\ 2 & -17 & 48 & -45 \\ -4 & 32 & -85 & 75 \\ 8 & -60 & 150 & -125 \end{pmatrix}$$

So far $r = s = t = u = 1$ in $\mathbf{T}_n$ gives a matrix with the nth row of Pascal's triangle in every row. A logical expectation is that this pattern will hold for higher n. To state this expectation more formally, the *binomial coefficient* formula comes into play.

$$\binom{n}{k} = C_k^n = \frac{n!}{k!(n-k)!} \tag{1.23}$$

This formula locates the kth term in the nth row of Pascal's triangle. Thus the $(n+1)^2$ with $r = s = t = u = 1$ will, if this pattern holds, be of the form

$$\begin{pmatrix} C_0^n & C_1^n & \cdots & C_{n-1}^n & C_n^n \\ C_0^n & C_1^n & \cdots & C_{n-1}^n & C_n^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_0^n & C_1^n & \cdots & C_{n-1}^n & C_n^n \\ C_0^n & C_1^n & \cdots & C_{n-1}^n & C_n^n \end{pmatrix} \tag{1.24}$$

The pattern in (1.24) is of use in finding larger $\mathbf{T}_n$. Another useful generalization expresses the perimeter of $\mathbf{T}_n$ using binomial coefficients as well.

$$\begin{pmatrix} t^n & C_1^n \cdot t^{n-1}u & C_2^n \cdot t^{n-2}u^2 & \cdots & C_{n-2}^n \cdot t^2u^{n-2} & C_{n-1}^n \cdot tu^{n-1} & u^n \\ rt^{n-1} & \bullet & \bullet & \cdots & \bullet & \bullet & su^{n-1} \\ r^2t^{n-2} & \bullet & \bullet & \cdots & \bullet & \bullet & s^2u^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r^{n-2}t^2 & \bullet & \bullet & \cdots & \bullet & \bullet & s^{n-2}u^2 \\ r^{n-1}t & \bullet & \bullet & \cdots & \bullet & \bullet & s^{n-1}u \\ r^n & C_1^n \cdot r^{n-1}s & C_2^n \cdot r^{n-2}s^2 & \cdots & C_{n-2}^n \cdot r^2s^{n-2} & C_{n-1}^n \cdot rs^{n-1} & s^n \end{pmatrix} \tag{1.25}$$

These two generalizations are applied in the effort to fill in $\mathbf{T}_4$.

$$i)\begin{pmatrix} t^4 & 4t^3u & 6t^2u^2 & 4tu^3 & u^4 \\ rt^3 & \bullet & \bullet & \bullet & su^3 \\ r^2t^2 & \bullet & \bullet & \bullet & s^2u^2 \\ r^3t & \bullet & \bullet & \bullet & s^3u \\ r^4 & 4r^3s & 6r^2s^2 & 4rs^3 & s^4 \end{pmatrix} \qquad ii)\begin{pmatrix} t^4 & 4t^3u & 6t^2u^2 & 4tu^3 & u^4 \\ rt^3 & t^2(3ru+st) & tu(3ru+3st) & u^2(ru+3st) & su^3 \\ r^2t^2 & rt(2ru+2st) & a & su(2ru+2st) & s^2u^2 \\ r^3t & r^2(ru+3st) & rs(3ru+3st) & s^2(3ru+st) & s^3u \\ r^4 & 4r^3s & 6r^2s^2 & 4rs^3 & s^4 \end{pmatrix}$$

Table 35: Steps in the process of finding $\mathbf{T}_4$

In *i* in table 35, end columns take the pattern of diminishing powers and the 4th row of Pascal's triangle provides scalars for the top and bottom rows. With nine unknowns now in the interior of *i*, this 5 x 5 version won't yield to simultaneous equations as readily as did the 4 x 4 $\mathbf{T}_3$. Another approach is to extrapolate, fill in all but the center point based on trends seen in the smaller matrices. Thus, the central elements in *ii*, save *a*, are educated guesses.

As a test, square *ii* in table 35 and call the result (not shown) $\mathbf{M}_4$. Now an equation such as, say, $\mathbf{M}_{4[3,1]} = \text{sqrt}(\mathbf{M}_{4[1,1]}) \cdot \text{sqrt}(\mathbf{M}_{4[5,1]})$ is solved for a in terms of r, s, t and u. The answer is $a = r^2u^2 + 4rstu + s^2t^2$. This, as in (1.26), completes the matrix.

$$\mathbf{T}_4 = \begin{pmatrix} t^4 & 4t^3u & 6t^2u^2 & 4tu^3 & u^4 \\ rt^3 & t^2(3ru+st) & tu(3ru+3st) & u^2(ru+3st) & su^3 \\ r^2t^2 & rt(2ru+2st) & r^2u^2+4rstu+s^2t^2 & su(2ru+2st) & s^2u^2 \\ r^3t & r^2(ru+3st) & rs(3ru+3st) & s^2(3ru+st) & s^3u \\ r^4 & 4r^3s & 6r^2s^2 & 4rs^3 & s^4 \end{pmatrix} \tag{1.26}$$

The central element in (1.26), introduces a novel term into the patterns. Until $r^2u^2 + 4rstu + s^2t^2$ appeared in $\mathbf{T}_{4[3,3]}$, it looked as if a general pattern for $\mathbf{T}_n$ might soon become evident. But it now seems likely that ever more elaborate combinations of r, s, t and u will emerge in the interiors of these matrices as they grow larger. So surely a few more, and maybe even a lot more of these $\mathbf{T}_n$ will need to be seen before we can, using this empirical approach, get a sense of what to expect and figure out how to formalize it.

To generate more of these matrices becomes in some ways more difficult as they grow larger, yet somewhat easier in other ways as the edge and other patterns become more familiar. E.g.,

$$\mathbf{T}_5 \;=\; \begin{pmatrix} t^5 & 5t^4u & 10t^3u^2 & 10t^2u^3 & 5tu^4 & u^n \\ t^4r & t^3(4ru+st) & t^2u(6ru+4st) & tu^2(4ru+6st) & u^3(ru+4st) & su^4 \\ t^3r^2 & rt^2(3ru+2st) & t(3r^2u^2+6rstu+s^2t^2) & u(r^2u^2+6rstu+3s^2t^2) & su^2(2ru+3st) & s^2u^3 \\ t^2r^3 & r^2t(2ru+3st) & r(r^2u^2+6rstu+3s^2t^2) & s(3r^2u^2+6rstu+s^2t^2) & s^2u(3ru+2st) & s^3u^2 \\ tr^4 & r^3(ru+4st) & r^2s(4ru+6st) & rs^2(6ru+4st) & s^3(4ru+st) & s^4u \\ r^5 & 5r^4s & 10r^3s^2 & 10r^2s^3 & 5rs^4 & s^n \end{pmatrix} \qquad (1.27)$$

$\mathbf{T}_5$ helps to clarify that numbers in rows 2 and n are predictable as 'Pascal partitions' of the number directly above/below. I.e., $C_k^n \;=\; C_{k-1}^{n-1} \;+\; C_k^{n-1}$. This gives the horizontal sides of the second perimeter in figure 6 below. (The vertical sides of the second perimeter (columns 2 and n) are even easier to predict, but with several options for their notation, they're left blank.)

$$\begin{pmatrix} t^n & C_1^n \cdot t^{n-1}u & C_2^n \cdot t^{n-2}u^2 & \cdots & C_{n-2}^n \cdot t^2u^{n-2} & C_{n-1}^n \cdot tu^{n-1} & u^n \\ rt^{n-1} & t^{n-2}(C_1^{n-1}ru + C_0^{n-1}st) & t^{n-3}u(C_2^{n-1}ru + C_1^{n-1}st) & \cdots & tu^{n-3}(C_{n-1}^{n-1}ru + C_{n-2}^{n-1}st) & u^{n-2}(C_n^{n-1}ru + C_{n-1}^{n-1}st) & su^{n-1} \\ r^2t^{n-2} & \bullet & \bullet & \cdots & \bullet & \bullet & s^2u^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r^{n-2}t^2 & \bullet & \bullet & \cdots & \bullet & \bullet & s^{n-2}u^2 \\ r^{n-1}t & r^{n-2}(C_0^{n-1}ru + C_1^{n-1}st) & r^{n-3}s(C_1^{n-1}ru + C_2^{n-1}st) & \cdots & rs^{n-3}(C_{n-1}^{n-1}ru + C_{n-2}^{n-1}st) & s^{n-2}(C_{n-1}^{n-1}ru + C_n^{n-1}st) & s^{n-1}u \\ r^n & C_1^n \cdot r^{n-1}s & C_2^n \cdot r^{n-2}s^2 & \cdots & C_{n-2}^n \cdot r^2s^{n-2} & C_{n-1}^n \cdot rs^{n-1} & s^n \end{pmatrix}$$

Figure 6: A tentative description of the second perimeter's top and bottom rows.

While a general formula may eventually be derived empirically, a derivation based on logical principles, one that provides a theoretical understanding of how the elements of $\mathbf{T}_n$ relate to one another in ways that don't change under exponentiation, seems the ultimate goal…

Problems:

Find the general form of $\mathbf{T}_6$, the 7 x 7 matrix that generates S^6.

Find the general form of $\mathbf{T}_n$, the $(n+1)^2$ matrix that generates S^n.

Addenda

Myriad other OEIS sequences (or candidates) can be found here… e.g., apply T_2^k, (k an index rather than exponent) = 1,2,3…, to $x^2 - x - 1$ coefficients, and clear fractions. Table 1 shows results for k = 1..20.

k =	a	b	c	$a+b+c$
20	100	120	–89	131
19	361	437	–319	479
18	81	99	–71	109
17	289	357	–251	395
16	64	80	–55	89
15	225	285	–191	319
14	49	63	–41	71
13	169	221	–139	251
12	36	48	–29	55
11	121	165	–95	191
10	25	35	–19	41
9	81	117	–59	139
8	16	24	–11	29
7	49	77	–31	95
6	9	15	–5	19
5	25	45	–11	59
4	4	8	–1	11
3	9	21	1	31
2	1	3	1	5
1	1	5	5	11

Table 1: T_2^k applied to the coefficients of $x^2 - x - 1$

The a column terms are A168077. (Note that this sequence is also generated as the square of a_n in A026741; we'll encounter this latter sequence again.) The b column in table 20 is A171621; dividing every 4th term by 4 gives A061037.The c column terms 5,1,1,–1,–11,–5,–31… are A229526. Note that the sum of a, b and c in a row n is the c coefficient in row n + 4 with the sign reversed (A229525).

In table 2, k of T_2^k takes fractional values; Coefficients are again (1 –1 –1); fractions in the table are cleared.

k =	a	b	c	$a+b+c$
$^1/_{13}$	13	689	8777	9479
$^1/_{12}$	6	294	3451	3751
$^1/_{11}$	11	495	5315	5821
$^1/_{10}$	5	205	1996	2206
$^1/_9$	9	333	2909	3251
$^1/_8$	4	132	1021	1157
$^1/_7$	7	203	1367	1577
$^1/_6$	3	75	430	508
$^1/_5$	5	105	497	607
$^1/_4$	2	34	127	163
$^1/_3$	3	39	107	149
$^1/_2$	1	9	16	25
1	1	5	5	11

Table 2: T_2^k applied to $x^2 - x - 1$ coefficients with fractional arguments for k

The a coefficients in table 2 (A026741) are the square roots of a coefficients in table 1. It's interesting that fractional arguments in k should have that effect. Such other mathematical relationships as may exist between the coefficients in these two tables are not so obvious…

One definition of a *Lucas number* (L_n) is $L_n = F_{n-1} + F_{n+1}$. Here is a Lucas-Fibonacci identity that popped out in the course of the exploration. If this identity in fact new, what will it take to prove it?

$$L_n F_j + F_{j-n} (-1)^{n+1} = F_{j+n}$$

References:

N. J. A. Sloane's Online Encyclopedia of Integer Sequences

Wolfram's Mathworld

Wikipedia