Borel summation of a family of divergent series

Peter Bala, Dec 27 2022

Introduction

The identity

$$\sum_{n=0}^{\infty} \frac{1}{n^n} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots = \int_0^1 \frac{dx}{x^x} \tag{1}$$

is known humorously as the Sophomore's dream [3]. See A073009. It is the case $a = b = t = 1$ of the more general result

$$\sum_{n=0}^{\infty} \frac{t^n}{(an+b)^{n+1}} = \int_0^1 \frac{x^{b-1}}{x^{tx^a}} \mathrm{d}x \quad a, b \in \mathbb{Z}_{>0}. \tag{2}$$

For a proof of (1) see [3]. The proof of (2) is exactly similar: write the integrand in (2) as $x^{b-1}\exp(-tx^a \log(x)) = \sum_{n=0}^{\infty}(-1)^n t^n x^{na+b-1} \log^n(x)/n!$, interchange the order of summation and integration, and then make the change of variable $x = \exp(-u/(an+b))$ to evaluate the integrals.

Watson [2, Section 5] proved the following result, originally stated by Ramanujan: the Borel sum of the divergent series $1 - 1^1 + 2^2 - 3^3 + 4^4 - \cdots$ is equal to $\int_1^{\infty} \frac{dx}{x^x}$.

We write Watson's result as

$$1 - 1^1 + 2^2 - 3^3 + 4^4 - 5^5 + \cdots = \int_1^{\infty} \frac{dx}{x^x} \quad \text{(Borel)}. \tag{3}$$

The numerical vaule of the integral is 0.7041699604.... See A245637.

The purpose of this note is to extend Watson's result and prove further Borel summability results such as

$$1^1 - 3^2 + 5^3 - 7^4 + 9^5 - 11^6 + \cdots = \int_1^{\infty} \frac{dx}{x^{x^2}} \quad \text{(Borel)}$$

and

$$2^1 - 5^2 + 8^3 - 11^4 + 14^5 - 17^6 + \cdots = \int_1^{\infty} \frac{dx}{x^{x^3}} \quad \text{(Borel)}.$$

The general result (Proposition 2, below) is

$$\sum_{n=0}^{\infty} (-1)^n (an+b)^n = \int_1^{\infty} \frac{x^{a-b-1}}{x^{x^a}} \mathrm{d}x \quad \text{(Borel)} \quad a, b \in \mathbb{Z}_{\geq 0}.$$

Borel summation along $\mathbb{R}^+$

Let $A(z)$ denote a formal power series

$$A(z) = \sum_{n=0}^{\infty} a(n) z^n.$$

Borel summation of $A(z)$ (in the direction of the positive real axis) is a method of assigning a value to the series $A(z)$, and involves three steps:

Step 1. Form the Borel transform of A defined as

$$\mathcal{B}\left(A(z)\right) = \sum_{n=0}^{\infty} \frac{a(n)}{n!} z^n.$$

Since the n-th coefficient of the series $\mathcal{B}(A(z))$ is smaller by a factor $n!$ than the corresponding coefficient of $A(z)$, the Borel transform $\mathcal{B}(A(z))$ may have a nonzero radius of convergence even if $A(z)$ does not. Assuming this is the case, we may sum the series in some open disc centred at $z = 0$.

Step 2. Suppose further that we can analytically continue this sum on the whole of the positive real axis $\mathbb{R}^+$. Denote the analytic continuation by $F(z)$.

Step 3. Find the Laplace transform $\mathcal{L}(F)$ of F:

$$\mathcal{L}(F)(z) = \int_0^{\infty} F(t) e^{-tz} \mathrm{d}t. \tag{4}$$

The existence of the improper integral requires exponential bounds on the growth of $F(x)$ for large real x. If the integral converges at $z_0 \in \mathbb{C}$, say to $\widetilde{A}(z_0)$, we say that the Borel sum of A converges at z_0, and write

$$\sum_{n=0}^{\infty} a(n) z_0^n = \widetilde{A}(z_0) \quad \text{(Borel)}.$$

Reversion of series

Watson's proof of (3) makes use of the identity

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \frac{n^n}{n!} \left(x \exp(x)\right)^n, \tag{5}$$

whose proof uses the Lagrange inversion formula. In order to generalise this identity we need the more general Lagrange-Bürmann formula for the reversion of power series, which we state in the following form:

Let $f(w)$ be an analytic function in a neighbourhood of 0 with $f(0) = 0$. Then there is a function $g(w)$, analytic in a neighbourhood of 0, which is the inverse of $f(w)$, that is, $f(g(w)) = g(f(w)) = w$. Let $H(w)$ be an arbitrary analytic function in a neighbourhood of 0. Then the coefficients in the power series expansion of the function $H(g(w))$ about the point $w = 0$ are given by

$$[w^n]\, H(g(w)) = \frac{1}{n}\left[t^{n-1}\right]\left(H'(t)\left(\frac{t}{f(t)}\right)^n\right). \tag{6}$$

(The particular case of this result when $H(w) = w$ is the Lagrange inversion formula.)

We generalise (5) as follows.

Proposition 1. *Let a and b be nonnegative integers. Then*

$$\frac{x^b}{1+ax^a} = \sum_{n=0}^{\infty}(-1)^n(an+b)^n\,\frac{(x\exp(x^a))^{an+b}}{n!} \tag{7}$$

in a neighbourhood of $x = 0$.

Proof. We split the proof into three cases.

(i) Case $a = 0$ is trivial.

(ii) Suppose $b = 0$. We have the identity [1, formula (14)]

$$\frac{1}{1+W(x)} = \sum_{n=0}^{\infty}(-1)^n\frac{n^n}{n!}x^n, \quad |x| < e^{-1}, \tag{8}$$

where the function $W(x)$, which satisfies $W(x)\exp(W(x)) = x$, is the principal branch of Lambert's W function.

Replacing x in (8) with $W^{-1}(x) = x\,\exp(x)$ gives

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n\frac{n^n}{n!}\,(x\exp(x))^n\,, \quad |x\exp(x)| < e^{-1},$$

and then replacing x with ax^a yields (7) with $b = 0$.

(iii) Suppose now a and b are both positive integers. Let $z = x\exp\,(x^a)$. Let $H(x) = \dfrac{x^b}{b}$. We apply the Lagrange–Bürmann formula to find $H(x)$ in terms of z. After a short calculation, we arrive at the result

$$\frac{x^b}{b} = \sum_{n=0}^{\infty}(-1)^n(an+b)^{n-1}\frac{z^{an+b}}{n!}\,. \tag{9}$$

Differentiating (9) with respect to x gives

$$\begin{aligned} x^{b-1} &= \sum_{n=0}^{\infty}(-1)^n(an+b)^n \frac{z^{an+b-1}}{n!}\frac{\mathrm{d}z}{\mathrm{d}x} \\ &= \sum_{n=0}^{\infty}(-1)^n(an+b)^n \frac{z^{an+b}}{n!}\frac{(1+ax^a)}{x} \end{aligned}$$

leading to the identity

$$\frac{x^b}{1+ax^a} = \sum_{n=0}^{\infty}(-1)^n(an+b)^n \frac{(x\exp(x^a))^{an+b}}{n!}$$

valid in some neighbourhood of $x=0$. □

The rational function on the left side of (7) is analytic throughout the complex plane, except for a finite number of poles, which all lie off the positive real axis since a is positive, and thus gives an analytic continuation of the series in (7) to an open set in $\mathbb{C}$ containing $\mathbb{R}^+ \cup \{0\}$.

We now have the ingredients to find the Borel sum of the alternating divergent series $\sum_{n=0}^{\infty}(-1)^n(an+b)^n$.

Proposition 2. *Let a, b be nonnegative integers. Then*

$$\sum_{n=0}^{\infty}(-1)^n(an+b)^n = \int_1^{\infty} \frac{x^{a-b-1}}{x^{x^a}}\mathrm{d}x \quad \text{(Borel)}.$$

Proof. The case $a=0$ is straightforward. Assume now a is a positive integer. We have

$$\begin{aligned} \sum_{n=0}^{\infty}(-1)^n(an+b)^n &= \sum_{n=0}^{\infty}(-1)^n \frac{(an+b)^n}{n!}\int_0^{\infty} t^n e^{-t}\mathrm{d}t \\ &= \int_0^{\infty}\left\{\sum_{n=0}^{\infty}(-1)^n\frac{(an+b)^n}{n!}t^n\right\}e^{-t}\mathrm{d}t \\ &= a\int_0^{\infty}\left\{\sum_{n=0}^{\infty}(-1)^n\frac{(an+b)^n}{n!}t^{an+b}\right\}t^{a-b-1}e^{-t^a}\mathrm{d}t \quad (t \to t^a)\,. \end{aligned}$$

Since ue^{u^a} increases steadily from 0 to ∞ as u increases from 0 to ∞, we have, on writing ue^{u^a} for t in the last integral,

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n(an+b)^n &= a\int_0^\infty \left\{\sum_{n=0}^{\infty}(-1)^n\frac{(an+b)^n}{n!}\left(ue^{u^a}\right)^{an+b}\right\}u^{a-b-1}e^{(a-b-1)u^a}e^{-u^a e^{au^a}}\,\mathrm{d}\left(ue^{u^a}\right) \\
&= a\int_0^\infty \left\{\frac{u^b}{1+au^a}\right\}u^{a-b-1}e^{(a-b-1)u^a}e^{-u^a e^{au^a}}e^{u^a}(1+au^a)\,\mathrm{d}u \\
&\qquad\qquad \text{(by Proposition 1)} \\
&= a\int_0^\infty u^{a-1}e^{(a-b)u^a-u^a e^{au^a}}\,\mathrm{d}u \\
&= \int_1^\infty \frac{x^{a-b-1}}{x^{x^a}}\,\mathrm{d}x
\end{aligned}$$

on making the change of variable $x = e^{u^a}$. □

References

[1] *R. M. Corless, D. J. Jeffrey, and D. E. Knuth* A sequence of series for the Lambert W function, Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation (Kihei, HI), pp. 197–204.

[2] *G. N. Watson,* Theorems stated by Ramanujan (VIII): , Journal of the London Mathematical Society, Volume s1-4, Issue 2, April 1929, Pages 82-86.

[3] *Wikipedia,* Sophomore's dream.