Asymptotic of sequences A161630, A212722, A212917 and A245265
(Václav Kotěšovec, published July 16 2014)

In the OEIS (On-Line Encyclopedia of Integer Sequences) published Paul D. Hanna in 2009 sequences A161630, A212722, A212917 and I added in 2014 sequence A245265, which can be generalized (for \( p \geq 1 \)) as the family of the sequences with an exponential generating function \( A(x) \), satisfies functional equation

\[
A(x) = \exp \left( \frac{x}{1 - x A(x)^p} \right)
\]
or as the sum

\[
a_n = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{n-k} \binom{1 + p \ast (n-k)}{k-1}
\]

**Theorem (V. Kotěšovec, July 15 2014):**
Asymptotic

\[
a_n \sim \frac{n^{n-1} p^{n-1+\frac{1}{p}} \left( 1 + 2 \ast \text{LambertW} \left( \frac{\sqrt{p}}{2} \right) \right)^{n+\frac{1}{2}}}{e^n 2^{2n+\frac{2}{p}} \text{LambertW} \left( \frac{\sqrt{p}}{2} \right)^{2+n+\frac{2}{p}} 1 + \text{LambertW} \left( \frac{\sqrt{p}}{2} \right)}
\]

**Proof:**
Following theorem by Edward A. Bender is (in case of implicit functions) very useful (for proof see [1], p.505 and also [4], p.469).

**Citation:** Edward A. Bender, "Asymptotic methods in enumeration" (1974), p.502, see [1]

**Theorem 5.** Assume that the power series \( w(z) = \sum a_n z^n \) with nonnegative coefficients satisfies \( F(z, w) = 0 \). Suppose there exist real numbers \( r > 0 \) and \( s > a_0 \) such that

(i) for some \( \delta > 0 \), \( F(z, w) \) is analytic whenever \( |z| < r + \delta \) and \( |w| < s + \delta \);

(ii) \( F(r, s) = F_s(r, s) = 0 \);

(iii) \( F_z(r, s) \neq 0 \), and \( F_{ww}(r, s) \neq 0 \); and

(iv) if \( |z| \leq r, |w| \leq s \), and \( F(z, w) = F_w(z, w) = 0 \), then \( z = r \) and \( w = s \).

Then

\[
a_n \sim \left( \frac{r F_z}{(2\pi F_{ww})^{1/2} n^{3/2} r^{-n}} \right),
\]

where the partial derivatives \( F_z \) and \( F_{ww} \) are evaluated at \( z = r, w = s \).

Bender's formula modified for exponential generating function is

\[
\frac{a_n}{n!} \sim \frac{1}{n^r} \frac{\sqrt{\frac{r F_z}{2\pi n F_{ww}}}}{e^{\frac{n}{r}} \sqrt{n}}
\]

\[
a_n \sim \frac{n^{n-1}}{e^n r^{n-1/2}} \sqrt{\frac{F_z}{F_{ww}}}
\]
Now we have the implicit function

\[ f(x, y) = e^{x - x y^p} - y \]

| \( F_z \) | \( \frac{\partial}{\partial x} f(x, y) \) | \( e^{x - x y^p} \) \\
|-----|-----------------|--------|
|     |                 | \((y^p - 1)^2\) \\
| \( F_w \) | \( \frac{\partial}{\partial y} f(x, y) \) | \( px^2 y^{p-1} e^{x - x y^p} \) \\
|     |                 | \((y^p - 1)^2\) - 1 \\
| \( F_{ww} \) | \( \frac{\partial}{\partial y} \frac{\partial}{\partial y} f(x, y) \) | \( \frac{px^2 y^{p-2} e^{x - x y^p}}{(y^p - 1)^4} (p(x^2 y^p (y^p - 1) - 1) + (xy^p - 1)^2) \) \\

\( r, s \), are roots of the system of equations

\[ e^{r - rs^p} - s = 0 \quad \frac{pr^2 s^{p-1} e^{r - rs^p}}{(rs^p - 1)^2} - 1 = 0 \]

From first equation we have

\[ r = \frac{1}{s^p + \frac{1}{\log (s)}} \]

and second equation can be reduced

\[ pr^2 s^p = (rs^p - 1)^2 \]

\[ p s^p (\log(s))^2 = 1 \]

\[ s = e^\left(\frac{2 \text{W}\left(\frac{x}{p}\right)}{p}\right) \]

where \( \text{W} \) is the \textbf{LambertW} function

\[ r = \frac{4 \left(\text{W}\left(\sqrt{p}/2\right)\right)^2}{p \left(1 + 2 \text{W}\left(\sqrt{p}/2\right)\right)} \]

The asymptotic is then

\[ a_n \sim \frac{n^{-1/2}}{\sqrt{e^{n^{-\lambda}}}} \frac{\sqrt{F_z}}{\sqrt{F_{ww}}} = e^{-n^{n^{-1} - n}} \frac{s^{2-p}(rs^p - 1)^2}{pr(p(r^2 s^p(s^p - 1) - 1) + (rs^p - 1)^2)} \]

and after simplification

\[ a_n \sim \frac{n^{-1} p^{n^{-1} + 1/2}}{e^n 2^{n^{2-n} + \frac{1}{2}} W(\sqrt{p}/2)^{2n^{2-n} + \frac{1}{2}}} \frac{1 + 2 * W(\sqrt{p}/2)^{n^{1/2}}}{1 + W(\sqrt{p}/2)} \]
Numerical verification (for $p=4$, sequence A245265)

```
Show[
ListPlot[
Table[Sum[n!*(1+p*(n-k))^((k-1)/k)*Binomial[n-1,n-k], {k, 0, n}]/
  (p^(n-1+1/p)*(1+2*LambertW[Sqrt[p]/2])^(n+1/2)*
   n^(n-1)/(Sqrt[1+LambertW[Sqrt[p]/2]]*E^n*2^(2*n+2/p)*
   LambertW[Sqrt[p]/2]^(2*n+2/p-1/2))/p -> 4, {n, 1, 100}]],
Plot[1, {n, 1, 100}, PlotStyle -> Red]]
```

References:


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