The Special Rational Sequence

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Back in 2005, I wrote a "thing"(for want of a better word) called *A Fraction Problem*. It looked at an interesting (at least, to me) sequence of rational numbers, a sequence I had worked with for about ten years, albeit initially with a singular term (the 4th). You can read the full file on the university pages here, but I'll summarise the contents here (and somewhat correct and simplify in places as well, since the original observations were a little scrambled).

Each value in the sequence was the fraction

$$\cfrac{1}{\cfrac{2}{\cfrac{3}{\frac{4}{\vdots}+\frac{4}{\vdots}}+\cfrac{3}{\frac{4}{\vdots}+\frac{4}{\vdots}}}+\cfrac{2}{\cfrac{3}{\frac{4}{\vdots}+\frac{4}{\vdots}}+\cfrac{3}{\frac{4}{\vdots}+\frac{4}{\vdots}}}}$$

which continued in a similar fashion down to n. Now, of course, this could be simplified in style to

$$\cfrac{1}{2\times\cfrac{2}{2\times\cfrac{3}{2\times\cfrac{4}{\vdots}}}}$$

which in turn could be simplified to the function definition

$$f:\mathbb{N}\longrightarrow\mathbb{Q}, n\longmapsto g(1,n) \text{ where } g:\mathbb{N}^2\longrightarrow\mathbb{Q}, (m,n)\longmapsto\begin{cases} n & \text{if } m=n \\ \dfrac{m}{2g(m+1,n)} & \text{otherwise}\end{cases}$$

...And that's where the research stopped.

The explanation for this is pretty silly: I didn't think there was anything more consequential about it, and didn't explore it any more. That remained the case until 2013, when I decided, in a moment of required patience, I decided to look back on the fractions. There was nothing more to it than that, but what caught my eye when I did made the whole exercise worth its measure. I'll give the first few terms below (and onto the next page):

$$n=1 \qquad 1=1$$

$$n=2 \qquad \frac{1}{4}=\frac{1}{2\times 2}$$

$$n=3 \qquad \cfrac{1}{2\times\cfrac{2}{2\times 3}}=\cfrac{1}{2\times\cfrac{1}{3}}$$

$$= \frac{3}{2} = \frac{1 \times 3}{2}$$

$n = 4$

$$\cfrac{1}{2 \times \cfrac{2}{2 \times \cfrac{3}{2 \times 4}}} = \cfrac{1}{2 \times \cfrac{2}{2 \times \cfrac{3}{8}}}$$

$$= \cfrac{1}{2 \times \cfrac{8}{3}}$$

$$= \frac{3}{16} = \frac{1 \times 3}{2 \times 2 \times 4}$$

$n = 5$

$$\cfrac{1}{2 \times \cfrac{2}{2 \times \cfrac{3}{2 \times \cfrac{4}{2 \times 5}}}} = \cfrac{1}{2 \times \cfrac{2}{2 \times \cfrac{3}{\cfrac{4}{5}}}}$$

$$= \cfrac{1}{2 \times \cfrac{2}{2 \times \cfrac{15}{4}}}$$

$$= \cfrac{1}{2 \times \cfrac{4}{15}}$$

$$= \frac{15}{8} = \frac{1 \times 3 \times 5}{2 \times 4}$$

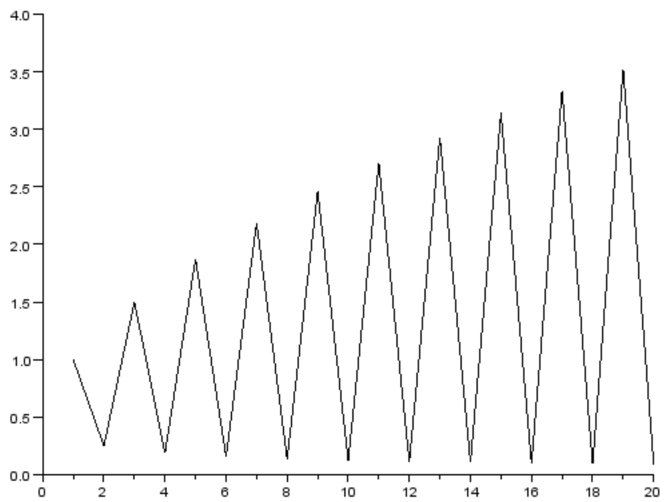
Now, at about term 5, we start to see a bit of a pattern emerge, albeit initially it seems in two pieces. The odd terms seem to be of the form $\frac{n!!}{(n-1)!!}$ while the even terms are of the form $\frac{(n-1)!!}{2 \cdot n!!}$.[1] With a little creative manipulation, the rational sequence can now be explained using one simple(ish) formula[2]:

$$\upsilon(n) = \left(\frac{(n-1)!!}{2^{\frac{1+(-1)^n}{2}} \cdot n!!} \right)^{(-1)^n}$$

Now that we have that out of the way, let's examine the pattern itself. I wo't deal so much with the terms themselves, not because they aren't interesting, but because the interesting things about them require more detail than I can provide. The graph, on the other hand, is a very interesting case indeed.

[1] The !! is the double factorial, calculated using the recursion $n!! = n \cdot (n-2)!!$ with $1!! = 0!! = 1$.

[2] The choice of upsilon (υ) in place of f is purely aesthetic, although this choice may result in these rationals being referred to as "upsilon numbers"; I really don't care either way as to their name, just as long as its nothing offensive.



I've cheated a little here, and used a program called Scilab to create this graph, using the following function[3]:

```
function y = upsilon(x)
    if ((int(x) < 1)) then error("Incorrect parameter type: Positive real matrix,
        vector or scalar expected.")
    end

    function m = df(n)
        if ((int(n) == 0) | (int(n) == 1)) then m = 1;
        else m = n * df(n - 2);
        end
    endfunction

    y = ones(x);

    for i = 1:size(x,1)
        for j = 1:size(x,2)
            y(i,j) = (df(x(i,j)-1)/(2^((1 + (-1)^x(i,j))/2) * df(x(i,j))))^((-1)^x(i,j));
        end
    end
endfunction
```

We immediately notice that there are two pathways to this, again: the values for when n is odd are divergent from zero, while the values for when n is even converge towards zero. So the question is: what does the sequence as a whole do? As n approaches infinity, would the pattern approach zero or infinity? Truth be told, I don't actually know, and both seem possible.

[3]Do not worry too much about the `error` function on the first line; this function was created to take a matrix as input, so it needed to test the matrix's values and make sure they are positive