A115032 and A240926, with an additional note on A246638 with A246639,
A246638 with A246640, A007805 with A246641 and A246642.

A geometrical application suggested by Kival Ngaokrajang.
Wolfdieter Lang, Aug 23 2014.

A touching circles problem considered by Kival Ngaokrajang (originally proposed under A240926 , later reused for a similar problem). See the link by Kival Ngaokrajang.

I) A115032: The touching circle sequence in the larger part of a circular disk with radius 5/4 (in some length units) bisected by a chord of length 2.

This sequence a(n) = A115032(n), n >= -1, with a(-1) = 1, appears in a nice geometrical problem with a sequence of touching circles considered by Kival Ngaokrajang (originally proposed under A240926). The input is a circle C with radius r = 5/4 (in some length units), a chord of length 2 such that the larger sagitta (perpendicular distance from the midpoint of the larger arc to the chord) has also length 2, and a circle C(0) with this larger sagitta as diameter, that is r(1) = 1. See the K. Ngaokrajang link for an illustration. The sequence of touching circles {C(n)}, n >= 0, in the left half of the disk of circle C is defined as follows: C(n) touches i) the cord, ii) the input circle C and iii) the circle C(n-1), for n>=1. It is clear that this circle sequence can be mirrored to the right hand side of the disk of circle C. It is also clear that there will actually be two solutions to this problem, because only C(1) (different from C(0)) will be uniquely defined, but the following circles could also fall back to previous ones: for example, C(2) could also be the input C(0). Therefore only the solution with a new (smaller) circle radius will be considered.
The input quantities are the radius R of C (the convenient input radius r =5/4 will be found later), the radius R(n) of circle C(n) and h, the length of the smaller sagitta for a chord of the length of the larger sagitta. A recurrence for R(n), or for the curvature 1/R(n), will be found, using the perpendicular distance of the mid point of circle C(n) to the larger sagitta, called x(n). If the midpoint of the chord is taken as origin, the x-axis along the chord to the right, and the y-axis along the sagitta to the top then the midpoint M(n) of C(n) has coordinates [-x(n), -y(n)] with x(0) = 0. From i) y(n) = R(n). From ii) x(n)^2 = h*(2*(R - R(n)) - h). From iii) x(n) = x(n-1) +2*sqrt(R(n)*R(n-1)).
A recurrence for R(n) follows by taking from ii) 2*h*R(n) = 2*R*h - h^2 - x(n)^2, eliminating x(n) with iii), then eliminating x(n-1) = sqrt(h*(2*(R-R(n-1)) -h)) with ii). From this equation follows by squaring

[4*sqrt(h*(2*(R-R(n-1)) -h))*sqrt(R(n)*R(n-1))]^2 = [-2*h*R(n) -2*h*(R - R(n-1)) -4*R(n)*R(n-1)]^2,

a quadratic equation for R(n) which has for R=1 and h =2/5 the two solutions:
R(n;+) = (9 - 5*R(n-1) + 2*R(n-1)*sqrt(5)*sqrt(4 - 5*R(n-1)))/(1 + 5*R(n-1))^2 and
R(n;-) = (9 - 5*R(n-1) - 2*R(n-1)*sqrt(5)*sqrt(4 - 5*R(n-1)))/(1 + 5*R(n-1))^2.
This produces with R(0) = 4/5: R(1) = [4/25, 4/25], R(2) = [4/5, 4/405], ... As mentioned above the second solutions are needed here. These examples suggest to use the scaled radii r(n) = (5/4)*R(n). For them the recurrence becomes:

r(n) = [9 - 4*r(n-1) - 4*sqrt(5)*sqrt(1-r(n-1))*r(n-1)] / (1 + 4*r(n-1))^2, n>=1, with r(0) =1.

Finally for the scaled curvatures or bends b(n) = 1/r(n) = 4/(5*R(n)) the recurrence is

b(n) = ((4 + b(n-1))^2)/(-4 + 9*b(n-1) - 4*sqrt(5)*sqrt(b(n-1)*(b(n-1) -1))), n >= 1, with b(0) = 1.

This is the sequence [1, 5, 81, 1445, ...] which looks like A115032(n-1), n >= 0 (with A115032(-1) = 1). This coincidence has been conjectured by Kival Ngaokrajang.
A crucial simplification occurs when the denominator and numerator is multiplied by
-4 + 9*b(n-1) + 4*sqrt(5)*sqrt(b(n-1)*(b(n-1) -1)). Then the new denominator cancels the old numerator. Thus,

b(n) = -4 + 9*b(n-1) + 4*sqrt(5)*sqrt(b(n-1)*(b(n-1) -1)) , n >= 1, with b(0) = 1.

The proof of the above mentioned conjecture can be accomplished as follows.
Clearly, this one step recurrence determines b(n) uniquely from the given input b(0). If we find a positive integer solution then it will be this unique solution. To find it we define
Y(n) := sqrt(b(n)*(b(n) - 1)/5) for n >= 0 with Y(0) = 0. That is a quadratic equation for b(n) solved by b(n;+) = (1 + sqrt(1 + 5*Y(n)^2))/2 and b(n;-) = (1 - sqrt(1 + 5*Y(n)^2))/2 , but because b(n) > 0 we only use b(n) = b(n;+).

b(n) = (1 + sqrt(1 + 5*Y(n)^2))/2, n >= 0.

Now all non-negative integer solutions can be taken from the Pell equation X^2 - 5*Y^2 =1, and then b(n) = (1 +X(n))/2. For the Pell equations see e.g., the reference O. Perron, Die Lehre von den Kettenbruechen, Teil I, Teubner, Stuttgart, 1954, 3. Auflage, Satz 3.18. p. 93. See also the on-line program for indefinite binary quadratic forms http://www.itp.kit.edu/~wl/BinQuadForm.html by the author WL.
Because the continued fraction for sqrt(5) = [2; repeat(4)] has a primitive period length k =1, there is an infinitude of non-negative solutions (X(n), Y(n)), n >=0, which is given by the numerators and denominators of the convergents, namely X(n) = A(2*n -1) and Y(n) = B(2*n-1) with A and B satisfying the recurrence q(n) = 4*q(n-1) + q(n-2) with inputs A(-1) = 1, A(0) = 2 and B(-1) = 0 and B(0) = 1. This leads to the identification X(n) = A023039(n) = T(n,9) = (S(n, 18) - S(n-2, 18)/2 = S(n, 18) - 9*S(n-1, 18), namely [1, 9, 161, ...], and Y(n) = A060645(n) = 4*S(n, 18), namely [0, 4, 72, ...], with Chebyshev's S- and T- polynomials (see the coefficient triangles A049310 and A053120, respectively). Therefore we have found positive integer solutions for b(n) which satisfy the original recurrence b(n) = -4 + 9*b(n-1) + 20*Y(n-1) with input b(0) =1 and the given Y(n-1) = 4*S(n-1, 18), namely

b(n) = (1+ S(n, 18) - 9*S(n-1 ,18))/2, n >= 0.

This produces therefore the solution for the scaled curvatures b(n) = 4/(5*R(n)) which has now been proven to coincide with A115032(n-1), with A115032(-1) = 1 (see a formula given there).
Therefore, in order to produce integer curvatures, one chooses for the input circle C the radius r =5/4 and for the cord the length 2*r(1) = (10/4)*4/5 = 2.

II) A240926: The touching circle sequence in the upper part of a circular disk with radius 5/4 (in some length units) bisected by a chord of length 2.

Similar to the treated case one can consider, as Kival Ngaokrajang did in the new A240926, the touching circle sequence in the left hand side of the upper part (minor) of the disk of a circle C with radius r=5/4 and a chord of length 2. A mirror sequence appears then on the right hand side.
The requirements for circles C'(n) of radius R'(n) are now to touch i) the cord, ii) the input circle C with radius R, and iii) the circle C'(n-1), for n>=1. Again a quadratic equation with two solutions will appear, and only one solution will be of interest. The midpoint of circle C'(n) has coordinates [-x'(n), y'(n)] (see above for the origin at the middle of the chord and the two axes) . The touching conditions are from i) y'(n) = R'(n), from ii) x'(n)^2 = 2*R'(n)*(h-2*R) + h*(2*R - h), and from iii) x'(n) = x'(n-1) +2*sqrt(R'(n)*R'(n-1)) (like in the above problem). Solving i) for R'(n), inserting x(n) from iii) , then replacing x(n-1) from ii), leads after writing the term with square-roots on one side, after squaring to the following equation:

(4*sqrt(2*R'(n-1)*(h-2*R) + 2*R*h - h^2)*sqrt(R'(n)*R'(n-1)))^2 -
(-2*(2*R-h)*R'(n) - 2*R'(n-1)*(h-2*R) - 4*R'(n)*R'(n-1))^2 = 0 .

This is a quadratic equation for R'(n) with two solutions, given for R=1 and h=2/5 by:

R'(n;+) = (6 - 5*R'(n-1) + 2*sqrt(5)*sqrt(1 - 5*R'(n-1)))*R'(n-1)/(4 + 5*R'(n-1))^2 and
R'(n;-) = (6 - 5*R'(n-1) - 2*sqrt(5)*sqrt(1 -5*R'(n-1)))*R'(n-1)/(4 + 5*R'(n-1))^2.

With R'(0) =1/5 this produces for R'(1) (4/25, 4/25) (only one solution on the left hand side of the disk) , for R'(2) (1/5, 4/45) , again the first solution is the radius of C'(0) and not the one looked for. Therefore we take the second solutions and rescale like above (we are in the same circular disk of C as above) r'(n) = (5/4)*R'(n), leading to the recurrence

r'(n) = (3-2*r'(n-1)-sqrt(5)*sqrt(1-4*r'(n-1)))*r'(n-1) / (2*(r'(n-1)+1)^2) with r'(0) =1/4.

For the scaled curvatures b'(n) = 1/r'(n) this becomes

b'(n) = 2*(1+b'(n-1))^2 / (3*b'(n-1) - 2 - sqrt(5)*sqrt((b'(n-1)-4)*b'(n-1)))   with b'(0) = 4.

Multiplying the denominator and numerator with 3*b'(n-1) - 2 + sqrt(5)*sqrt((b'(n-1)-4)*b'(n-1)) leads to b'(n) = (3*b'(n-1) - 2 + 5*sqrt((b'(n-1) - 4)*b'(n-1)/5))/2, with b'(0) = 4.

Like above we show that the unique solution of this recurrence is one with positive integers. For this define Y'{n) = sqrt((b'(n) - 4)*b'(n)/5). This quadratic equation implies b'(n) = 2 + sqrt(4 + 5*Y'(n)^2) (the positive solution for b(n) has been chosen). All non-negative integer solutions of the Pell equation X'(n)^2 - 5*Y'(n)^2 = +4 are known to be given by (see the above mentioned on-line program) the two proper sequences of solutions (7, 3), (123, 55), (2207, 987) , ... and (3, 1), (47, 21), (843, 377), … , as well as the improper solutions (2, 0), (18, 8), (322, 144), ... Altogether these are the solutions X'(n) = A005248(n) = L(2*n) = S(n, 3) - S(n-2, 3), namely [2, 3, 7, 18, 47, 123, ...], and
Y'(n) = A001906(n) = F(2*n) = S(n-1, 3), namely [0, 1, 3, 8, 21, 55, ...] , with Chebyshev's S-polynomials already encountered above, the Fibonacci numbers F(n) = A000045(n) and the Lucas numbers L(n) = A000032(n) . Therefore, by the same argument as above, the solution of the original recurrence for b'(n) is given by the positive integers

b'(n) = 2 + L(2*n) = 2 + S(n, 3) - S(n-2, 3) = 2 + 2*S(n, 3) - 3*S(n-1, 3), n >= 0.

This is the sequence A240926(n), n>= 0, [4, 5, 9, 20, 49, 125, 324, 845, 2209, 5780, 15129, ...].

Notes added Sep 05 2014

The final formula in part I) had a misprint (twice S(n, 18)) which has been corrected, and some words have been changed.

Proofs for A246638 with A246639, A246638 with A246640, A007805 with A246641 and A246642.

The complement relative to the large disk (radius 5/4) of the circular disks found in parts I) and II) above can be considered, in a next step, to find new sequences for curvatures of touching circles.
This will be an application of the famous touching circle theorem by Descartes (and Vieta and Steiner). See, e.g., http://mathworld.wolfram.com/DescartesCircleTheorem.html, or https://en.wikipedia.org/wiki/Descartes'_theorem. Given three pairwise touching circles with curvatures (or bends) b(j), j=1, 2, 3, the quadratic equation to find the curvature for the two circles, each touching the given three ones is:

(sum(b(j), j=1..4))^2 = 2*sum(b(j)^2, j=1..4).

The solutions b(4;+) and b(4;-) are:

b(4;+) = b(1) + b(2) + b(3) + 2*sqrt(b(1)*b(2) + b(1)*b(3) + b(2)*b(3)),
b(4;-) = b(1) + b(2) + b(3) - 2*sqrt(b(1)*b(2) + b(1)*b(3) + b(2)*b(3)).

Note that one has to use negative curvature for a circle circumscribing other circles. For degenerate circles (straight lines) one takes b = 0. This formula does obviously not work if the three given circles touch in one point, because then any other circle through this point will do the job, but there would be at most two solutions from the formula. For the following discussion we use the fact that the formula works also if one of the circles degenerates into a straight line (provided there is not only one common touching point with the two other circles).
Note that Descartes' theorem does not apply to parts I) and II) above because the chord does not touch the large circle.

III a) A246638 with A246639.

In this section we consider touching circles in the smaller part of the circular disk with radius 5/4, bisected by a chord of length 2, which touch i) the circle with radius 5/4 and ii) two neighboring touching circles of part II).
These circles will appear in the illustration, given by Kival Ngaokrajang as a link in A240926, in the upper regions between the disks in the upper part and the large circle. One could proceed as above but we use now Descartes' theorem with b(1) = -4/5 (the large circle circumscribes the circles in the upper part), b(2) = b'(n) = A240926(n), b(3) = b'(n+1) = A240926(n+1), n>=0, from part II).
For example, the first circle (n=0) has curvature b''(0) = b(4;+) = -4/5 + 4 + 5 + 2*sqrt((-4/5)*(4 + 5) + 4*5) = (41+16*sqrt(5))/5 = (25 + 32*phi)/5 approx. 15.35541753, where phi is the golden section phi = (1 + sqrt(5))/2. This is not an integer in the quadratic number field Q(sqrt(5)) (it would be such an integer if the radii would be scaled, for example, by a factor of 1/5). The corresponding radius is 1/b''(0), approx. 0.06512359550.
The other solution is b(4;-) = (57 - 32*phi)/5, approx. 1.044582474, and it gives the curvature of the circle which touches

the large circle in the lower part and the two circles in the upper part from below, and is not of interest here. The general sequence for the b(4;+) type solution turns out to be

b"(n) = A"(n) + (B"(n)/5)*phi, with A"(n) = 2 + 3*(S(n, 3) - S(n-1, 3)) and B"(n) = 4*(3 + 5*(S(n,3) - S(n-1,3))), n >= 0,

This is the sequence [5+(32/5)*phi, 8+(52/5)*phi, 17+(112/5)*phi, 41+(272/5)*phi, 104+(692/5)*phi, 269+(1792/5)*phi, ...].

S(n, 3) - S(n-1, 3) = A001519(n+1), A"(n) = A246638(n) and B"(n)/4 = A246639(n), n >= 0.

Proof: b(n) = -4/5 + A(n) + A(n+1) + 2*sqrt((-4/5)*(A(n) + A(n+1)) + A(n)*A(n+1)), n >= 0, with A(n) = A240926(n) = 2 + 2*S(n, 3) - 3*S(n-1, 3) (see part II) above). This is compared with the claim b(n) = A"(n) + (B"(n)/5)*phi with the given A"(n) and B"(n) formulas. Write the square root term on one side and square. The claim is then that 4*((-4/5)*(A(n) + A(n+1)) + A(n)*A(n+1)) = (-(-4/5 + A(n) + A(n+1)) + A"(n) + (B"(n)/5)*(1 + sqrt(5))/2)^2. The following identities for the S-polynomials are useful here. The well known Cassini-Simson type identity is (i) S(n, x)*S(n-2, x) = -1 + S(n-1, x)^2, for n>=0, with S(-2, x) = -1 and S(-1, x) = 0. Together with the three term recurrence S(n, x) = x*S(n-1, x) - S(n-2, x) this implies (for x=3) (ii) S(n, 3)*S(n-1, 3) = (-1 + S(n, 3)^2 + S(n-1, 3)^2)/3 and also iii) S(n+1, 3)*S(n, 3) = (8/3)*S(n, 3)^2 - (1/3)*S(n-1, 3)^2. Only the recurrence for S(n+1, 3) and identity (ii) are needed here. The latter in order to eliminate the mixed term S(n, 3)*S(n-1, 3). This shows that the expanded equation is true.

III b) A246638 with A246640.

Here we consider touching circles in the upper part of the circular disk with radius 5/4, bisected by a chord of length 2, which touch i) the chord and ii) two touching circles of part II). These (small) circular disks would appear in the illustration, given by Kival Ngaokrajang as a link in A240926, in the regions between the touching circles in the smaller part of the bisection and the chord.
One could proceed as above in parts I) and II) or use again Descartes' theorem with b(1) = 0 (the chord as degenerate circle), b(2) = b'(n) = A240926(n), b(3) = b'(n+1) = A240926(n+1), n>=0, from part II).
For example, the first circle (n=0) has curvature b_3(0) = b(4;+) = 0 + 4 + 5 +2*sqrt(4*5) = 9 + 4*sqrt(5) = 5 + 8*phi, approx. 17.94427191, where phi is the golden section phi = (1 + sqrt(5))/2. This is an integer in the real quadratic number field Q(sqrt(5)). The corresponding radius is 1/b_3(0), approx. 0.055728092. The other solution is b(4;-) = 9 - 4*sqrt(5) = 13 - 8*phi, approx. 0.055728092. This is the reciprocal of b(4;+) = b_3(0). This large circle lies on the left hand side of the two given ones and touches the chord far left from above (see the special case considered in the Wikipedia link). It will not be of interest here.
The general sequence for the b(4;+) type solution turns out to be

b_3(n) = A"(n) + B_3(n)*phi, with A"(n) = 2 + 3*(S(n, 3) - S(n-1, 3)) = A246638(n), like above, and B_3(n) = 4*(1 + S(n,3) - S(n-1,3)) = 4*(1 +A001519(n+1)) = 4*A246640(n), n >= 0.

This is the sequence [5+8*phi, 8+12*phi, 17+24*phi, 41+56*phi, 104+140*phi, 269+360*phi, ...]. The curvatures b_3(n) are integers in the real quadratic number field Q(sqrt(5)).

The proof is done like above in part a). From Descartes' theorem b_3(n) = A(n) + A(n+1) + 2*sqrt(A(n)*A(n+1)), n >= 0, with A(n) = A240926(n) = 2 + 2*S(n, 3) - 3*S(n-1, 3) (see part II) above). This is compared with the claim b_3(n) = A"(n) + B_3(n)*phi with the given A"(n) and B_3(n) formulas. Write the square root term on one side and square. The claim is then that 4*A(n)*A(n+1) = (-(A(n) + A(n+1)) +A"(n) + B_3(n)*(1+sqrt(5))/2)^2 written in terms of the S-polynomials. Use for S(n+1, 3) the recurrence and for S(n, 3)*S(n-1, 3) identity (ii) given above in the proof of part a). This will end the proof.

IV a) A007805 with A246641.

This section describes the touching circles and chord problem in the larger part of a circular disk with radius 5/4 bisected by a chord of length 2. We consider the curvature of the small circles touching i) the chord and ii) two touching circles with curvatures b(n) and b(n+1) from part I). These (small) circles would appear in the illustration, given by Kival Ngaokrajang as a link in A240926, in the lower part of the bisection in the regions between the touching circles and the chord.
Descartes' theorem is applied with b(1) = 0 (the chord as degenerate circle), b(2) = b(n) = A115032(n-1) and b(3) = b(n+1) = A115032(n), n >= 0, from part I).
For example, the first circle has curvature b_4(0) = b(4;+) = 0 + 1 + 5 + 2*sqrt(1*5) = 6 + 2*sqrt(5) = 4*(1 + phi), approx.

10.47213595. This is an integer in the real quadratic number field Q(sqrt(5)). The corresponding radius is 1/b_4(0), approx. 0.09549150286. The other solution is b(4;-) = 6 - 2*sqrt(5) = 4*(2 - phi), approx. 1.527864046 with radius 1/b(4;-) approx. 0.6545084968. This circle touches the chord, the circle with radius 1 from the inside, and the circle with radius 1/5 from the outside, and will not be of interest here.
The general sequence for the b(4;+) type solution turns out to be

b_4(n) = 4*(A_4(n) + B_4(n)*phi), with A_4(n) = S(n, 18) - S(n-1, 18) = A007805(n), and B_4(n) = (1 + S(n, 18) - S(n-1, 18))/2 = (1 +A007805(n))/2 = A246641(n), n >= 0.

This is the sequence [4 + 4*phi, 68+36*phi, 1220+612*phi, 21892+10948*phi, 392836+196420*phi, ...]. The curvatures b_4(n) are integers in Q(sqrt(5)).

The proof is done like above by comparing b_4(n) = b(n) + b(n+1) +2*sqrt(b(n)*b(n+1)), where b(n) = A115032(n-1) = (1 + S(n,18) - 9*S(n-1, 18))/2 , with 4*(A_4(n) + B_4(n)*(1+sqrt(5))/2) written in terms of S-polynomials. Use the recurrence S(n+1, 18) = 18*S(n, 18) - S(n-1, 18) and the identity (ii) from above but with (x=18) S(n, 18)*S(n-1, 18) = (-1 + S(n, 18)^2 + S(n-1, 18)^2)/18.

IV b) A246642.

This section describes a touching circles problem in the larger part of a circular disk with radius 5/4 bisected by a chord of length 2. The curvature of the small circles touching (i) the outer circle with radius 5/4 and (ii) two touching circles with curvatures b(n) and b(n+1) from part I) is considered here. These (small) circles would appear in the illustration, given by Kival Ngaokrajang as a link in A240926, in the larger part of the bisection between the outer circle and the touching circles from part I).
Descartes' theorem is applied with b(1) = -4/5 (because the outer circle circumscribes the other ones), b(2) = b(n) = A115032(n-1) and b(3) = b(n+1) = A115032(n), n>=0, from part I).
For example, the first circle (n=0) has curvature b_5(0) = b(4;+) = -4/5 + 1 + 5 + 2*sqrt((-4/5)*(1 + 5) + 1*5) = (2/5)*(13 + sqrt(5)) = (4/5)*(6 + phi), approx. 6.094427191. This is not an integer in the real quadratic number field Q(sqrt(5)) (it would become one after scaling the radii with any positive integer multiple of 1/5). The corresponding radius is 1/b_5(0), approx. 0.1640843296. The other solution is b(4;-) = (2/5)*(13 - sqrt(5)) = (4/5)*(7 - phi), approx. 4.305572809, with radius 1/b(4;-) approx. 0.2322571338 . This corresponds to the circle above the circle with curvature 5, reaching also into the upper part, and it will not be of interest here.
The general sequence for the b(4;+) type solution turns out to be

b_5(n) = (4/5)*(A_5(n) + B_5(n)*phi) with A_5(n) = (1 +5*(Snx(n, 18)-Snx(n-1, 18)) = 4 + 2*A246642(n) and B_5(n) = (-3 + 5*(Snx(n, 18)-Snx(n-1, 18)))/2 = A246642(n), n >= 0.

This is the sequence [(4/5)*(6+phi), (4/5)*(86+41*phi), (4/5)*(1526+761*phi), (4/5)*(27366+13681*phi), ...].

The proof is done like above by comparing b_5(n) = -4/5 + b(n) + b(n+1) + 2*sqrt((-4/5)*(b(n) + b(n+1)) + b(n)*b(n+1)), where b(n) = A115032(n-1) = (1 + S(n,18) - 9*S(n-1, 18))/2 with (4/5)*(A_5(n) + B_5(n)*(1+sqrt(5))/2) written in terms of S-polynomials. Use the recurrence S(n+1, 18) = 18*S(n, 18) - S(n-1, 18) and the identity (ii) with x=18 from part a) above. This is all what is needed for the proof.

Note added, Sep 16 2014

Some typos have been corrected.

For the cases III) and IV) one computes the coordinates of the centers of the circles in a Cartesian coordinate system where the intersection points of the chord with the big circle (radius 5/4) are [-1,0] and [+1,0].

III a) The center of circle C"(n) with radius r"(n)=1/b"(n) has coordinates [-x"(n), y"(n)], n >= 0. From the touching condition of C"(n) with circle C'(n) of radius r'(n)=1/b'(n) from part II) one has i) (1/b'(n) + 1/b"(n))^2 = (x"(n) - x'(n))^2 + (y"(n) - 1/b'(n))^2, and from the touching of C"(n) and C'(n+1) one has ii) (1/b'(n+1) + 1/b"(n))^2 = (x"(n) - x'(n+1))^2 + (y"(n) - 1/b'(n+1))^2. These two equations can be use to compute two solutions for x"(n) and two for y"(n). Keep only the larger one of the y"(n) solutions, and put this into the condition obtained from the touching of C"(n) and the big circle which is iii) x"(n) = sqrt((5/4 - 1/b"(n))^2 - (y"(n) - (5/4 - 1/2))^2) (the positive root). For example, for n=0 one will find the two y''(0) solutions .4023146068, 0.04052184768 (Maple 10 digits). Take the first one to find x''(0) = 0.2758679739. Thus the coordinates of the center of C''(0) are approximately [-0.2758679739, 0.4023146068]. The radius r''(0) = 0.06512359550. The general quadratic equation for y''(n) will not be recorded here because its coefficients are fairly complicated.

III b) Here y_3(n) = r_3(n) = 1/ b_3(n) because C_3(n) touches the chord. C_3(n) touches C'(n) from part II) leading to i) (1/b_3(n) + 1/b'(n))^2 = (1/b'(n) -1/b_3(n))^2 + (x_3(n) - x'(n))^2. One takes the larger solution x_3(n) = x'(n) + 2/sqrt(b_3(n)*b'(n)) . The other condition is ii) (1/b_3(n) + 1/b'(n+1))^2 = (1/b'(n+1) -1/b_3(n))^2 + (x_3(n) - x'(n+1))^2 from C_3(n) touching C'(n+1). The smaller solution for x_3(n) will then have to coincide automatically with the found x_3(n). The center of C_3(n) is then [-x_3(n), 1/b_3(n)]. For example, for n=0 one finds the center [-0.2360679775, 0.05572809000], and the radius is 0.09549150286 (Maple 10 digits).

IV a) The center of the circle C_4(n) has coordinates denoted by [-x_4(n), -y_4(n)], n >= 0. Here y_4(n) = r_4(n) = 1/ b_4(n) because C_4(n) touches the chord from below. C_4(n) touches C(n) from part I) with b(n) and x(n) = sqrt(h*(2*(R - 1/b(n)) - h)) given there. This means i) x_4(n) = x(n) + 2/sqrt(b_4(n)*b(n)) (only the larger value is of interest because C_4(n) touches the left part of C(n). The other condition is similar but with b(n+1) and x(n+1) , but there the smaller solution has to be taken which then coincides with the one just found. For example, for n=0 the center has coordinates [-0.6180339888,-0.09549150286] and the radius is r_4(n) = 0.09549150286 (Maple 10 digits).

IV b) The center of circle C_5(n) with radius r_5(n) = 1/ b_5(n) has coordinates [-x_5(n), -y_5(n)], n >= 0. C_5(n) touches the big circle with radius 5/4, hence (similar like above in the case III a)) x_5(n) = sqrt((5/4 - 1/b_5(n))^2 - (y_5(n) - (5/4 - 1/2))^2) (the positive root). The other two touching conditions lead to two solutions for y_5(n), and here the larger solution is taken because C_5(n) touches C(n) and C(n+1) from below. The two x_5(n) solutions are not of interest, because the found y_5(n) is plucked into the x_5(n) equation just given. These two touching conditions are i) (1/b_5(n) + 1/b(n))^2 = (x_5(n) - x(n))^2 + (y_5(n) - 1/b(n))^2, and ii) (1/b_5(n) + 1/b(n+1))^2 = (x_5(n) - x(n+1))^2 + (y_5(n) - 1/b(n+1))^2. For example, for n=0 one finds the center of C_5(0) to be at [-1.061976090,-0.5232410335], and the radius is r_5(0) = 0.1640843296 (Maple 10 digits).

For these small circles see the figures in the W. Lang link "Figures for various touching circle problems" found under A240926.

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