ON THE NUMBER OF CORONAS OF A FIXED RHOMBUS WITH CHARACTERISTICS OF n–FOLD SYMMETRY IN $\mathbb{R}^2$

L. E. JEFFERY

Abstract. An algorithm is presented for constructing the irregular triangular array A233332 in *The On–line Encyclopedia of Integer Sequences*.

1. definitions and introduction

Throughout this note, let n be an integer, $n \geq 2$, and let $k \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

Definition 1. *Let R_n denote the set of $\lfloor \frac{n}{2} \rfloor$ rhombi in which the kth rhombus $r_{n,k} \in R_n$ has interior angles about its vertices (or corners) given by the pair $\left(\frac{k\pi}{n}, \frac{(n-k)\pi}{n}\right)$.*

Definition 2. *Let T be any tiling of the Euclidean plane constructed from tiles of R_n which fit together edge-to-edge and vertex-to-vertex without gaps or overlaps. For any tile $t \in T$, for $m > 0$, define the recurrence*

$$C_m(t) = C_{m-1}(t) \cup \{\text{all tiles of } T \text{ that intersect the boundary of } C_{m-1}(t)\},$$

with initial condition

$$C_0(t) = t.$$

For the tiling T and the tile $t \in T$, the patch of tiles $C_M(t)$, $M \geq 0$, is called the Mth corona *of t. Equivalently, starting with any tile $r \in R_n$ fixed in the plane, by juxtaposing enough tiles from R_n we can* compose *a corona $C_M(r)$ of r of any order M by tessellation. Figure 1 illustrates the first corona of a fixed tile in a tiling.*

Definition 3. *For any $r \in R_n$ fixed in the plane, a disjoint union of r with any four tiles $t_1, t_2, t_3, t_4 \in R_n$ each fitting edge-to-edge and vertex-to-vertex with a distinct edge of r is called a* candidate. *If no tiles overlap in a candidate, then that*

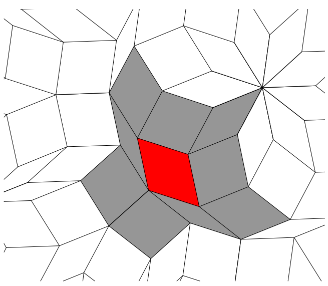

Figure 1. Part of a rhombus tiling showing a typical first corona: the shaded tiles completely surround or enclose the boundary of the red tile.

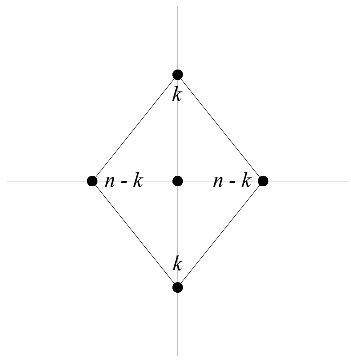

FIGURE 2. Generalized rhombus $r_{n,k}$ in preferred orientation centered in the plane, with interior angle coefficients shown.

*candidate is called an r–*core*; otherwise that candidate is rejected (since no corona of r can be constructed from it).*

The following problems in the theory of tiles seem to have not been addressed before in the literature:

Problem 1. *Fix $r_{n,k} \in R_n$ in the plane. For $m = 1, 2, \ldots$, in how many ways can $r_{n,k}$ be extended to an mth corona $C_m(r_{n,k})$ of $r_{n,k}$ using tiles from R_n?*

Problem 2. *From Problem 1, in how many ways can $r_{n,k}$ be so extended if isometries different from the identity are not counted?*

These problems are hard enough that here we are able to consider only the simplest cases. To that end, the remainder of this note is devoted to describing an algorithm for constructing the irregular triangular array A233332 in *The On–Line Encyclopedia of Integer Sequences*. We ultimately derive an expression for the general entry of A233332 and hence a solution (albeit a partial one) of Problem 1, for $m = 1$. It is not yet known whether or not any column of A233332 satisfies a recurrence relation.

2. An algorithm for constructing A233332

A fixed orientation of $r_{n,k} \in R_n$ in the plane (Definition 3) can of course be purely arbitrary, but to simplify construction we will fix $r_{n,k}$ centered at the origin $(0, 0)$ with its vertices incident with the points $\left(\pm \sin\left(\frac{k\pi}{2n}\right), 0\right)$ on the horizontal axis and $\left(0, \pm \cos\left(\frac{k\pi}{2n}\right)\right)$ on the vertical axis, that is, so that $r_{n,k}$ is bisected by both axes, as in Figure 2. Accordingly, the coefficient k (Definition 1) is always associated with vertices of $r_{n,k}$ which fall on the vertical axis.

Label the vertices of $r_{n,k}$ with the numbers $1, 2, 3, 4$ and mark the edges of $r_{n,k}$ with arrows consecutively in the counter–clockwise direction, as illustrated in Figure 3a. (The arrows are only for the purpose of instruction.) Let v_l denote the vertex of $r_{n,k}$ with index (or label) $l \in \{1, 2, 3, 4\}$.

For any n and k, the number of candidates is equal to $(n-1)^4$, with the number of possible $r_{n,k}$–cores generally being considerably less than $(n-1)^4$. Therefore, we have to consider all candidates to determine which of them are $r_{n,k}$–cores. Let $N = 0, 1, \ldots, (n-1)^4 - 1$, and associate each index N with a distinct candidate,

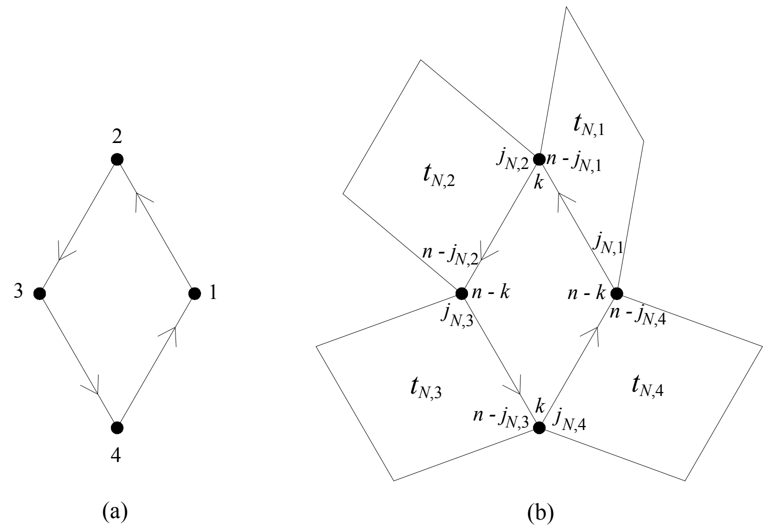

FIGURE 3. (a) Generalized rhombus $r_{n,k}$ with labeled vertices v_l and with edges marked with arrows; (b) the tile indices $j_{N,l}$ and their complements $n - j_{N,l}$ in their proper places in a construction of the Nth candidate.

as follows. Define the $(n-1)^4 \times 4$ array $J = (j_{N,l})$ with entries $j_{N,l}$, $l = 1, 2, 3, 4$, given by

$$j_{N,l} = 1 + \operatorname{mod}\left(\left\lfloor \frac{N}{(n-1)^{4-l}} \right\rfloor, n-1 \right).$$

The $j_{N,l}$ are called *tile indices*. The rows of J run through all ordered quadruples of tile indices from the set $\{1, 2, \ldots, n-1\}$ in lexicographical order. Row N of J represents the Nth candidate. The Nth candidate is then constructed as follows.

Let $t_{N,1}, t_{N,2}, t_{N,3}, t_{N,4} \in R_n$ be the four tiles (Definition 3) juxtaposed to $r_{n,k}$ in such a way that $t_{N,l}$ shares the edge of $r_{n,k}$ that has the arrow directed away from v_l and such that $j_{N,l} \in \{1, 2, \ldots, n-1\}$ is the interior angle coefficient of the angle $\frac{j_{N,l}\pi}{n}$ between the two edges of $t_{N,l}$ which meet at vertex v_l of $r_{n,k}$, where the angle is taken in the counter–clockwise direction about v_l. For illustration, the tile indices $j_{N,l}$ are marked on the tile $t_{N,l}$ near the vertex v_l, and the complementary indices $n - j_{N,l}$ are marked on $t_{N,l}$ near their associated vertices, in Figure 3b. Now the tile indices of the Nth candidate need to be tested, as follows.

Referring to Figure 3b, for $N = 0, 1, \ldots, (n-1)^4 - 1$, define the index functions

$$\begin{aligned} a(k; N, 1) &= 2n - (j_{N,1} + (n-k) + (n - j_{N,4})) = -j_{N,1} + k + j_{N,4}; \\ a(k; N, 2) &= 2n - (j_{N,2} + k + (n - j_{N,1})) = -j_{N,2} - k + n + j_{N,1}; \\ a(k; N, 3) &= 2n - (j_{N,3} + (n-k) + (n - j_{N,2})) = -j_{N,3} + k + j_{N,2}; \\ a(k; N, 4) &= 2n - (j_{N,4} + k + (n - j_{N,3})) = -j_{N,4} - k + n + j_{N,3}. \end{aligned}$$

For $N = 0, 1, \ldots, (n-1)^4 - 1$, define the flag functions

$$F(k, N) = \begin{cases} 1, & \text{if } a(k; N, l) \geq 0,\ l = 1, 2, 3, 4; \\ 0, & \text{otherwise.} \end{cases}$$

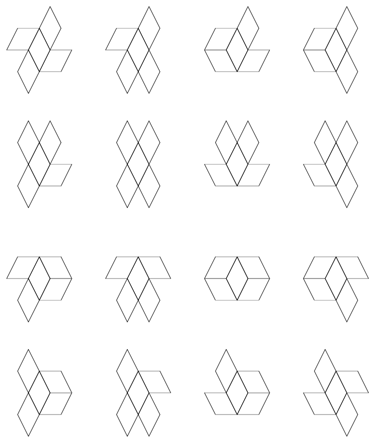

FIGURE 4. The sixteen $r_{3,1}$–cores arranged in a square array.

Then $F(k, N) = 1$, if and only if tiles do not overlap in the Nth candidate, that is, if and only if row N of J corresponds to an $r_{n,k}$–core. If $F(k, N) = 1$, then for the range of $a(k; N, l)$ we have $0 \leq a(k; N, 1) \leq 2n - k - 2$, for each $l \in \{1, 2, 3, 4\}$.

Let the symbol $[x^\alpha]\, f(x)$ denote the coefficient of x^α in the series $f(x)$. Suppose $F(k, N) = 1$. Denote by $S_{N,l}$ the untiled $r_{n,k}$–core "sector" of unit radius with interior angle $\frac{a(k;N,1)\pi}{n}$ taken counter–clockwise about v_l, $l \in \{1, 2, 3, 4\}$. For each l, there is a bijection that sends ordered tile indices of the arrangements of tiles of R_n which tessellate $S_{N,l}$, each tile of which is also vertex–incident with v_l, to the compositions of $a(k; N, l) > 0$ in which no part exceeds $n - 1$,[1] such that the tile indices and parts are ordered identically, each tile index corresponds to a part of equal magnitude, and conversely (proof is easy, so we omit it). Hence the number of extensions of the $r_{n,k}$–core with index N to a first corona of $r_{n,k}$ is equal to

$$\prod_{l=1}^{4} \left(\left[x^{a(k;N,l)} \right] \frac{1 - x}{1 - 2x + x^n} \right),$$

where the coefficient $\left[x^{a(k;N,l)}\right]$ is equal to the number of compositions of $a(k; N, l)$ in which no part exceeds $n - 1$. Now let A233332(n, k) denote the entry in row n and column k of array A233332. Then, to complete the algorithm, we can write

$$\text{A233332}(n, k) = \sum_{\substack{N \in \left\{0, \ldots, (n-1)^4 - 1\right\} \\ F(k,N)=1}} \prod_{l=1}^{4} \left(\left[x^{a(k;N,l)} \right] \frac{1 - x}{1 - 2x + x^n} \right),$$

for the number of first coronas of the rhombus $r_{n,k} \in R_n$ fixed in the plane.

Although the number of $r_{n,k}$–cores has to be $A(n, k) = \sum_{N=0}^{(n-1)^4 - 1} F(k, N)$, we conjecture that the columns of A satisfy $\sum_{h=0}^{5} (-1)^h \binom{5}{h} A(n + 5 - h, k) = 0$, for appropriate initial conditions. Finally, Figure 4 illustrates the sixteen $r_{3,1}$–cores from which can be counted a total of 83 possible first coronas of $r_{3,1}$.

E-mail address: `lejeffery2@gmail.com`

[1] Recall that the number of compositions of 0 is defined to be equal to 1, by convention.