SWINGING ORBITALS

A planar orbital system is a family of concentric circles in a plane divided into $n$ sectors. An orbital is a closed path consisting of arcs on these circles such that at each boundary of a sector the path jumps to the next inner or outer circle. One exception is allowed: if $n$ is odd the path may continue on the same circle, but just once.

After fixing one circle as the central circle there are three types of orbitals: a high orbital that never goes below the central circle, a low orbital that never goes above the central circle, and a swinging orbital that is neither a high nor a low orbital.

The diagrams below show all orbitals over 5 and 6 sectors. Actually each orbital system shows two orbitals: the one drawn in green and its dual, in orange. Dynamically, a particle moving along an orbital would start in a counterclockwise direction, except for the dual which would start in a clockwise direction.

In the diagrams, the kernel of an orbital system is white if the orbital is a high orbital and dark if it is a low orbital. Light gray signals a swinging orbital. The number of high orbitals is the same as the number of low orbitals.

<table>
<thead>
<tr>
<th>A056040</th>
<th>all orbitals</th>
<th>1, 1, 2, 6, 6, 30, 20, 140, 70, 630</th>
</tr>
</thead>
<tbody>
<tr>
<td>A057977</td>
<td>low (high) orbitals</td>
<td>1, 1, 1, 3, 2, 10, 5, 35, 14, 126, 42</td>
</tr>
<tr>
<td>A232500</td>
<td>swinging orbitals ($n \geq 2$)</td>
<td>$-1, -1, 0, 0, 2, 10, 10, 70, 42, 378$</td>
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</tbody>
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We can summarize as follows: The swinging factorial numbers A056040 count all orbitals over $n$ sectors, the extended Catalan numbers A057977 count the low and the high orbitals, respectively, and A232500 counts the swinging orbitals over $n$ sectors (nonpositive values indicate that there exist none).
Figure 1 – Orbitals over 5 sectors
Figure 2 – Orbitals over 6 sectors