11884. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA and Tudorel Lupu, Decebal High School, Constanța, Romania.* Let f be a real-valued function on $[0, 1]$ such that f and its first two derivatives are continuous. Prove that if $f(1/2) = 0$, then

$$\int_0^1 (f''(x))^2\,dx \geq 320 \left(\int_0^1 f(x)\,dx \right)^2.$$

11885. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Prove that

$$\sum_{p=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n)^4 + ((m+n)(m+p))^2} = \frac{3}{2}\zeta(3) - \frac{5}{4}\zeta(4).$$

Here ζ denotes the Riemann zeta function.

11886. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Suppose $n \geq 3$, and let $y_1, \dots, y_n$ be a list of real numbers such that $2y_{k+1} \leq y_k + y_{k+2}$ for $1 \leq k \leq n-2$. Suppose further that $\sum_{k=1}^n y_k = 0$.

Prove that

$$\sum_{k=1}^{n} k^2 y_k \geq (n+1) \sum_{k=1}^{n} k y_k,$$

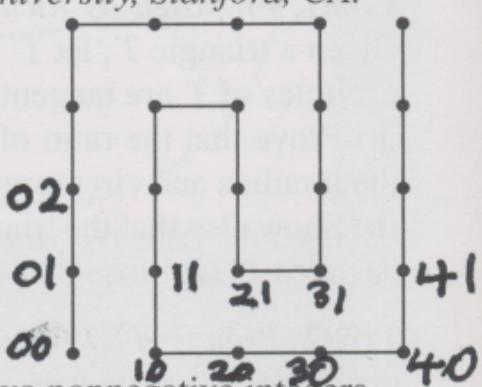
and determine when equality holds.

SOLUTIONS

A220952

A Twisted Enumeration of the Positive Integers

11733 [2013, 854]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Let $V = \{0, 1, 2, 3, 4\}^2$. Say that nonnegative integers a and b are *adjacent* when their base-5 expansions $\cdots a_2 a_1 a_0$ and $\cdots b_2 b_1 b_0$ satisfy the condition that if $i > j \geq 0$ and $(a_i, a_j) \neq (b_i, b_j)$, then (a_i, a_j) and (b_i, b_j) are consecutive in the path through V shown at right (horizontal coordinate listed first). Thus, for example, 0 is adjacent to 1. Similarly 48 (expansion 143_5) is adjacent to 47 (expansion 142_5) and 73 (expansion 243_5).



Need to prefix base-5 expansions with ∞ many zeros

(a) Prove that every positive integer is adjacent to exactly two nonnegative integers.
(b) Prove that with this definition of adjacency, the nonnegative integers form a path $\langle x_0, x_1, x_2, \dots \rangle$ starting with $x_0 = 0$.
(c) Explain how to compute efficiently from n the number x_n that comes n steps after 0, and determine $x_{1,000,000}$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. We first define a *raster path* P_d from 1^d to 3^d in $\{1, 2, 3\}^d$. For $d = 0$ the path is the single vertex given by a 0-tuple. Denote by aP the path given by prepending a to every vertex of a path P. For $d \geq 1$, the path P_d is the concatenation of $1P_{d-1}$, $2P'_{d-1}$, and $3P_{d-1}$, where P' denotes the reverse of P. By induction on d, every step along the path moves one digit up or down by 1. Call a step an $i|j$ step if it changes i to j or j to i in one position.

To find the nth vertex of P_d (starting from 0), write n in base 3, replace the kth digit by its difference from 2 if the number of 1s to the left is odd, and then increment each digit by 1. When two vertices differ only in position k, having 1 and 2 there, they are adjacent via a $1|2$ step in P_d if and only if all entries to the right are 3. When in position k they have 2 and 3, they are adjacent via a $3|2$ step in P_d if and only if all entries to the right are 1. Because P_d has 1^d at the beginning and 3^d at the end, it follows by induction on d that $1|2$ steps alternate with $3|2$ steps.

We next define a *double raster path* D_d from 0^d to 4^d in $\{0, 1, 2, 3, 4\}^d$. Begin with the path in $\{0, m, 4\}^d$ obtained from P_d by replacing every 1 with 0, every 2 with m, and every 3 with 4. Steps become $0|m$ steps or $4|m$ steps, alternating. Turn D_d into a path through $\{0, 1, 2, 3, 4\}^d$ as follows: When entering an element v of $\{0, m, 4\}^d$ with j entries equal to m, expand v into a copy of P_j in the coordinates equal to m, leaving the other coordinates fixed. If we enter v by a $0|m$ step, then we follow P_j from 1^j to 3^j. If we enter v by a $4|m$ step, then we follow P_j' from 3^j to 1^j. For example, the state $0m04m$ follows $0m044$ and expands to $03043, 03042, 03041, 02041, 02042, 02043, 01043, 01042, 01041$.

We summarize important properties of double raster paths, all of which can be verified by induction on d. Exactly two of properties (3)–(6) apply at each vertex other than 0^d and 4^d (where exactly one applies), governing the steps entering and leaving that vertex. Left and right are relative to the coordinate that changes in a step.

(1) All steps change a single coordinate to an adjacent value.
(2) The raster path is followed forward in the blocks arising from elements whose number of copies of m is odd and backwards in those where it is even.
(3) A vertex is involved in a $0|1$ step when each entry to the right is 4 and each entry to the left is 0, 1, or 4.
(4) A vertex is involved in a $1|2$ step when each entry to the right is 0, 3, or 4.
(5) A vertex is involved in a $2|3$ step when each entry to the right is 0, 1, or 4.
(6) A vertex is involved in a $3|4$ step when each entry to the right is 0 and each entry to the left is 0, 3, or 4.

For example, when $d = 4$ the vertex 2233 with leftmost entry 2 cannot satisfy **(3)** or **(6)**, so its transitions must involve 2; its neighbors are 2133 and 2232.

A position cannot first change away from 0 until each entry to the right is 4. Thus the first 5^{d-1} steps of the double raster path D_d are a copy of D_{d-1} through vertices with leading coordinate 0. Hence the double raster paths of all dimensions combine to give a single path D_∞ through $\{0, 1, 2, 3, 4\}^\infty$.

We now return to the given problem. Each edge in the graph shown corresponds to changing the entry in one position to a neighboring value. An edge of V that changes a vertical coordinate from 0 to 1 has horizontal coordinate 0, 1, or 4. An edge of V that changes a horizontal coordinate from 0 to 1 has vertical coordinate 4. Thus two integers that differ in a single digit where one has a 0 and the other a 1 are adjacent if and only if all digits to the left are 0, 1, or 4 and all digits to the right are 4. Comparing this to **(3)**, we see that they are adjacent if and only if they are consecutive in D_∞. Analyzing the other three cases similarly and comparing them to **(4)**, **(5)**, and **(6)**, we see that nonnegative integers are adjacent if and only if they are consecutive in D_∞. This establishes parts **(a)** and **(b)** of the problem.

For part **(c)**, let y_k be the string of symbols from $\{0, m, 4\}$ corresponding to the vertex x_k. If k has d digits in base 5, then y_k will have d characters (since D_∞ starts with D_d). If the leading digit of k is 4, say $k = 4 \cdot 5^{d-1} + k'$, then y_k will be given by prepending a 4 to $y_{k'}$ (with leading zeroes added if necessary to make $y_{k'}$ of length $d-1$), since the last segment of D_d is $4D_{d-1}$. If the leading digit is 1, 2, or 3, then let

$s = \lfloor (k - 5^d)/3 \rfloor < 5^{d-1}$. Now y_k will be given by prepending m to the complement (swapping symbols 0 and 4) of y_s, since the middle section of D_d is mD'_{d-1} and every block has three times as many vertices as the corresponding block of D_{d-1} due to the extra m. Running this calculation backwards, we can find the least j such that $y_j = y_k$. Hence we know how far into the raster path k is. Specifically, take the string y_k and swap the symbols 0 and 4 whenever there is an odd number of symbols m to the left. Call the resulting sequence $z_{d-1} \cdots z_0$. Define $k_0 = 0$ and

$$k_{j+1} = \begin{cases} k_j & \text{if } z_j = 0; \\ 3k_j + 5^j & \text{if } z_j = m; \\ k_j + 4 \cdot 5^j & \text{if } z_j = 4. \end{cases}$$

Now k_d is the least term in the block. From **(2)** we know whether the raster path is followed forwards or backwards, so we can compute the values for these coordinates as described in the opening paragraph and use them to find x_k.

For the specific case $k = 1000000$, we compute as follows. Letting an overbar denote complementation, and m^k to denote k consecutive copies of m, we obtain

$$\begin{aligned} y_{1000000} &= m\overline{y_{203125}} = m^2 y_{41666} = m^3\overline{y_{8680}} = m^4 y_{1851} = m^5\overline{y_{408}} \\ &= m^6 y_{94} = m^7\overline{y_{23}} = m^7 0\overline{y_3} = m^7 0m. \end{aligned}$$

We then calculate $k_1 = 1$, $k_2 = 21$, $k_3 = 88$, $k_4 = 389$, $k_5 = 1792$, $k_6 = 8501$, $k_7 = 41128$, $k_8 = 201509$, and $k_9 = 995152$. Thus, the first step in this block occurs at index 995152, and $x_{1000000}$ is the 4848th term in this block. We follow the raster path backwards in this block, because it has eight symbols m. Writing $4848 = 20122120_3$ in base 3 and complementing every digit with an odd number of 1s to its left gives 20100120. Hence the 4848th step of the raster path P_8 is 31211231 and the 4848th step counting backwards from the end is the complement 13233213. Hence $x_{1000000} = 132332103_5 = 667778$.

Also solved by Y. J. Ionin, O. P. Lossers (Netherlands), and the proposer. Parts (a) and (b) also solved by TCDmath Problem Group (Ireland).

A Powerful Equation

11734 [2013, 854]. *Proposed by Vahagn Aslanyan, Yerevan State University Yerevan, Armenia.* Find all lists (a, k, m, n) of positive integers such that

$$a^{m+n} + a^n - a^m - 1 = 15^k.$$

Solution by GCHQ Problem Solving Group, Cheltenham, U. K. There are two such lists, $(2, 1, 2, 2)$ and $(4, 1, 1, 1)$.

We require $15^k = (a^m + 1)(a^n - 1)$. We show first that 15 cannot divide both factors. If $3 \mid a^m + 1$, then $a \equiv 2 \pmod 3$ and m is odd. If $3 \mid a^n - 1$, then $a \equiv 2 \pmod 3$ requires n even.

If $5 \mid a^m + 1$, then m odd requires $a \equiv 4 \pmod 5$.

Since n is even, we have $a^n - 1 = (a^{n/2} - 1)(a^{n/2} + 1)$. These factors differ by 2, so neither 3 nor 5 can divide both. Since $15 \mid a^n - 1$, $n \geq 4$, so neither factor can be 1. Hence we have two cases.

Case 1: $a^{n/2} - 1 = 3^r$ *and* $a^{n/2} + 1 = 5^s$, *with* $r, s \geq 1$. If $s = 1$, then $a^{n/2} = 4$, and $a \equiv 4 \pmod 5$ requires $a = 4$ and $n = 2$. This yields $15 \mid 4^m + 1$, which fails since $4^m + 1$ is 2 or 5 modulo 15. If $s > 1$, then by Mihăilescu's theorem (Catalan's conjecture) $a^{n/2}$ is not a proper power, so $n = 2$. This yields $a = 1 + 3^r$, which contradicts $a \equiv 2 \pmod 3$.