A MODIFIED ENGEL EXPANSION FOR THE QUADRATIC IRRATIONAL $(\mathbf{p-1})\sqrt{p^2-1}-(\mathbf{p}^2-\mathbf{p}-\mathbf{1})$

PETER BALA

Lucas [Lu] showed that the quadratic irrational $\frac{1}{2}\left(p-\sqrt{p^2-4}\right)$, $p \geq 3$ an integer, has a predictable Engel expansion $[e_1, e_2, e_3, ...]$ given by $e_1 = p$ and $e_{n+1} = e_n^2 - 2$ for integer $n \geq 1$. We define a modified Engel expansion for a positive real number and show that the quadratic irrational $(p-1)\sqrt{p^2-1} - (p^2-p-1)$, $p \geq 2$ an integer, has the modified Engel expansion $[2, a_1, a_2, 2, a_3, a_4, ..., 2, a_n, a_{n+1}, ...]$, where $a_1 = p$, $a_{2n} = a_{2n-1} + 1$ and $a_{2n+1} = 2a_{2n-1}^2 - 1$ for $n \geq 1$.

1. The Engel expansion

Let x be a real number in the half-open interval $(0, 1]$. The Engel expansion of x is the unique non-decreasing sequence $[e_1, e_2, e_3, ...]$ of positive integers such that

$$x = \frac{1}{e_1} + \frac{1}{e_1 e_2} + \frac{1}{e_1 e_2 e_3} + \cdots .$$

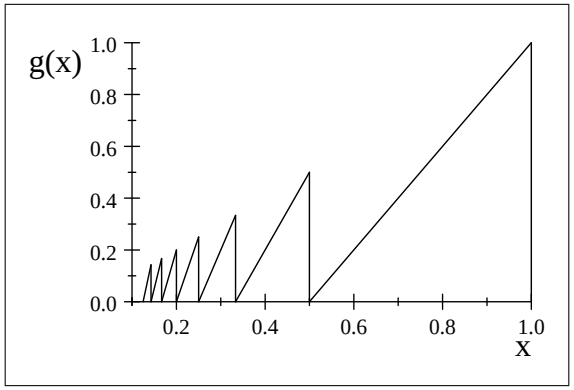
The terms e_n in the Engel expansion of x are obtained from the iterates of the map $g(x) : (0, 1] \mapsto (0, 1]$, defined as

$$g(x) = x\left(1 + \mathrm{floor}\left(\frac{1}{x}\right)\right) - 1,$$

by means of the formula

$$e_n = 1 + \mathrm{floor}\left(\frac{1}{g^{(n-1)}(x)}\right).$$

Here $g^{(n)}(x) = g(g^{(n-1)}(x))$ denotes the n-th iterate of $g(x)$, with the convention that $g^{(0)}(x) = x$.
The graph of the map $g(x)$ is shown below. The graph is piecewise linear with discontinuities at $x = \frac{1}{n}, n = 2, 3, ...$ The graph is a sawtooth shape with the 'teeth' decreasing in size to the left.



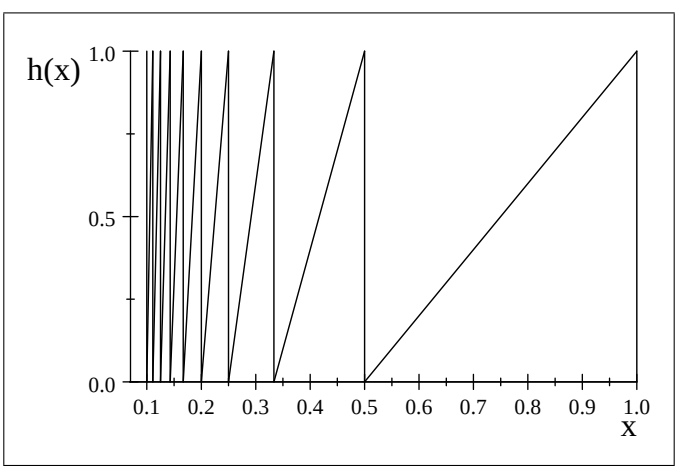
In the next section we will introduce a modification of the mapping $g(x)$. By studying the iterates of this modified map we will be lead to a modified Engel expansion for a real number.

2. The harmonic sawtooth map $h(x)$

We modify the map $g(x)$ to give a new mapping $h(x)$ of the half-open interval (0,1] by defining

$$\begin{aligned} h(x) &= \operatorname{floor}\left(\frac{1}{x}\right) g(x) \\ &= x \operatorname{floor}\left(\frac{1}{x}\right)\left(1+\operatorname{floor}\left(\frac{1}{x}\right)\right) - \operatorname{floor}\left(\frac{1}{x}\right). \end{aligned} \tag{1}$$

The mapping $h(x)$ has recently been studied by Crowley [2] who calls it the harmonic sawtooth map. The graph of $h(x)$ is shown below. The graph has a sawtooth shape, as for $g(x)$, but now the 'teeth' have constant height equal to 1. The rational points $x = \frac{1}{n}, n = 2, 3, ...$ are discontinuities for $h(x)$.



In section 3 we will define the modified Engel expansion of x in terms of the iterates $h^{(n)}(x)$ of the harmonic sawtooth map $h(x)$. Before doing this we need to establish some properties of these iterates.

Lemma 2.1. *The n-th iterate of $h(x)$ takes the form*

$$h^{(n)}(x) = a(n)x - b(n), \tag{2}$$

where $a(n)$ and $b(n)$ are sequences of positive integers.

PROOF. Straightforward induction on n. ■

Lemma 2.2. *The sequences $a(n)$ and $b(n)$ satisfy the recurrence equations*

$$a(n+1) = m(m+1)a(n) \tag{3}$$

$$b(n+1) = m(m+1)b(n) + m, \tag{4}$$

for $n \geq 0$, where $m = \mathrm{floor}\left(\frac{1}{h^{(n)}(x)}\right)$.

PROOF. The half-open intervals $(\frac{1}{n+1}, \frac{1}{n}]$, $n = 1, 2, ...$, cover (0,1]. Suppose $h^{(n)}(x)$ lies in the half-open interval $(\frac{1}{m+1}, \frac{1}{m}]$, so $m = \mathrm{floor}\left(\frac{1}{h^{(n)}(x)}\right)$. Then by definition (1) of the map $h(x)$ we find

$$\begin{aligned} h^{(n+1)}(x) &= h(h^{(n)}(x)) \\ &= m(m+1)a(n)x - m(m+1)b(n) - m \\ &= a(n+1)x - b(n+1). \end{aligned}$$

Thus

$$a(n+1) = m(m+1)a(n) \text{ and } b(n+1) = m(m+1)b(n) + m.$$

■

Corollary. *$a(n)$ forms an increasing sequence. Moreover*

$$a(n) \geq 2a(n-1) \geq 2^n.$$

PROOF. Immediate from the recurrence (3) since $m = \mathrm{floor}\left(\frac{1}{h^{(n)}(x)}\right)$ is a positive integer.
■

Lemma 2.3.

$$x = \lim_{n\to\infty} \frac{b(n)}{a(n)}.$$

PROOF.

$$x - \frac{b(n)}{a(n)} = \frac{a(n)x - b(n)}{a(n)} \leq \frac{1}{2^n},$$

using the previous corollary and recalling that $h^{(n)}(x) = a(n)x - b(n)$ belongs to $(0, 1]$. ■

3. The modified Engel expansion

We are now ready to define the modified Engel expansion of a number.

Definition. *Let $x \in (0, 1]$. Let $h(x) = x\,floor\left(\frac{1}{x}\right)\left(1 + \mathrm{floor}\left(\frac{1}{x}\right)\right) - floor\left(\frac{1}{x}\right)$ denote the harmonic sawtooth map. The modified Engel expansion of x is the sequence $[E_1, E_2, E_3, ...]$ of positive integers given by*

$$E_1 = 1 + \mathrm{floor}\left(\frac{1}{x}\right) \tag{5}$$

$$E_n = \mathrm{floor}\left(\frac{1}{h^{(n-2)}(x)}\right)\left\{1 + \mathrm{floor}\left(\frac{1}{h^{(n-1)}(x)}\right)\right\} \quad \text{for } n \geq 2, \tag{6}$$

where $h^{(n)}(x) = h(h^{(n-1)}(x))$ denotes the n-th iterate of the map $h(x)$, with the convention that $h^{(0)}(x) = x$.

Theorem 3.1.
(a) $x \in (0, 1]$ has a representation as an infinite series of unit fractions

$$x = \frac{1}{E_1} + \frac{1}{E_1E_2} + \frac{1}{E_1E_2E_3} + \cdots. \tag{7}$$

(b) For each $n \geq 1$ we have the finite series representation

$$x = \frac{1}{E_1} + \frac{1}{E_1E_2} + \cdots + \frac{1}{E_1 \ldots E_n} + \frac{1}{E_1 \ldots E_n} \frac{h^{(n)}(x)}{\mathrm{floor}\left(\frac{1}{h^{(n-1)}(x)}\right)}. \tag{8}$$

PROOF.
(a) We begin by converting the sequence $\frac{b(n)}{a(n)}$ of rational approximations to x in Lemma 2.3 into a telescoping series.

$$\frac{b(n)}{a(n)} = \frac{b(1)}{a(1)} + \left(\frac{b(2)}{a(2)} - \frac{b(1)}{a(1)}\right) + \cdots + \left(\frac{b(n)}{a(n)} - \frac{b(n-1)}{a(n-1)}\right). \tag{9}$$

It will turn out that the terms of this series are unit fractions. In anticipation of this result we define a sequence $c(n)$ by setting

$$\frac{1}{c(1)} = \frac{b(1)}{a(1)} \tag{10}$$

$$\frac{1}{c(n)} = \frac{b(n)}{a(n)} - \frac{b(n-1)}{a(n-1)} \quad [n \geq 2]. \tag{11}$$

So (9) becomes the series expansion

$$\frac{b(n)}{a(n)} = \frac{1}{c(1)} + \frac{1}{c(2)} + \cdots + \frac{1}{c(n)}. \tag{12}$$

We now show that

$$c(n) = E_1 \ldots E_n. \tag{13}$$

Firstly, from the definition (1) of the map $h(x)$ we find

$$\begin{aligned} \frac{1}{c(1)} &= \frac{b(1)}{a(1)} = \frac{\operatorname{floor}\left(\frac{1}{x}\right)}{\operatorname{floor}\left(\frac{1}{x}\right)\left(1+\operatorname{floor}\left(\frac{1}{x}\right)\right)} \\ &= \frac{1}{1+\operatorname{floor}\left(\frac{1}{x}\right)}. \end{aligned}$$

Thus

$$c(1) = 1 + \operatorname{floor}\left(\frac{1}{x}\right) = E_1.$$

Next, using the recurrences from Lemma 2.2 we have for $n \geq 2$

$$\begin{aligned} \frac{1}{c(n)} &= \frac{b(n)}{a(n)} - \frac{b(n-1)}{a(n-1)} \\ &= \frac{m(m+1)b(n-1)+m}{m(m+1)a(n-1)} - \frac{b(n-1)}{a(n-1)}, \quad \text{where } m = \operatorname{floor}\left(\frac{1}{h^{(n-1)}(x)}\right), \\ &= \frac{1}{(m+1)a(n-1)}. \end{aligned}$$

Thus

$$c(n) = (m+1)a(n-1), \quad \text{where } m = \operatorname{floor}\left(\frac{1}{h^{(n-1)}(x)}\right). \tag{14}$$

Hence using (3)

$$\begin{aligned} c(n) &= (m+1)m'(m'+1)a(n-2), \quad \text{where } m' = \text{floor}\left(\frac{1}{h^{(n-2)}(x)}\right), \\ &= (m+1)m'c(n-1) \quad \text{using (14)} \\ &= E_n c(n-1) \\ &= E_n E_{n-1} \dots E_1, \end{aligned}$$

and (13) has been proved.

The series expansion (12) now becomes

$$\frac{b(n)}{a(n)} = \frac{1}{E_1} + \frac{1}{E_1 E_2} + \cdots + \frac{1}{E_1 \dots E_n} \tag{15}$$

and hence by Lemma 2.3

$$\begin{aligned} x &= \lim_{n \to 0} \frac{b(n)}{a(n)} \\ &= \frac{1}{E_1} + \frac{1}{E_1 E_2} + \cdots + \frac{1}{E_1 \dots E_n} + \cdots \end{aligned}$$

and the proof of part (a) of the theorem is complete.

(b) Now (2) says

$$h^{(n)}(x) = a(n)x - b(n).$$

It follows that

$$\begin{aligned} x &= \frac{b(n)}{a(n)} + \frac{h^{(n)}(x)}{a(n)} \\ &= \frac{1}{E_1} + \frac{1}{E_1 E_2} + \cdots + \frac{1}{E_1 \dots E_n} + \frac{h^{(n)}(x)}{a(n)} \quad \text{by (15)} \\ &= \frac{1}{E_1} + \frac{1}{E_1 E_2} + \cdots + \frac{1}{E_1 \dots E_n} + \frac{h^{(n)}(x)\left(1 + \text{floor}\left(\frac{1}{h^{(n)}(x)}\right)\right)}{c(n+1)} \quad \text{by (14)} \\ &= \frac{1}{E_1} + \frac{1}{E_1 E_2} + \cdots + \frac{1}{E_1 \dots E_n} + \frac{h^{(n)}(x)}{E_1 \dots E_n \text{floor}\left(\frac{1}{h^{(n-1)}(x)}\right)}, \end{aligned}$$

where in the final step we have used (13) and the definition of E_{n+1}. ■

One consequence of part (b) of Theorem 3.1 is that if we approximate x by truncating the modified Engel expansion of x to n terms

$$x \approx \frac{1}{E_0} + \frac{1}{E_0 E_1} + \cdots + \frac{1}{E_0 \ldots E_n},$$

then the error made is less than or equal to the final term.

Example Calculate the first few terms of the modified Engel expansion of $\log(2)$.

The sequence of iterates $h^{(n)}(\log(2))$ and the associated rational approximations to $\log(2)$ begin

n	$h^n(\log(2))$	$\frac{b(n)}{a(n)}$
1	$2\log(2) - 1$	$\frac{1}{2}$
2	$12\log(2) - 8$	$\frac{8}{12}$
3	$144\log(2) - 99$	$\frac{99}{144}$
4	$288\log(2) - 199$	$\frac{199}{288}$
5	$576\log(2) - 399$	$\frac{399}{576}$

Converting the sequence of rational approximations into a telescoping series we obtain

$$\begin{aligned}
\log(2) &= \frac{1}{2} + \left(\frac{8}{12} - \frac{1}{2}\right) + \left(\frac{99}{144} - \frac{8}{12}\right) + \left(\frac{199}{288} - \frac{99}{144}\right) + \left(\frac{399}{576} - \frac{199}{288}\right) + \cdots \\
&= \frac{1}{2} + \frac{1}{6} + \frac{1}{48} + \frac{1}{288} + \frac{1}{576} + \cdots \\
&= \frac{1}{2} + \frac{1}{(2\text{x}3)} + \frac{1}{(2\text{x}3\text{x}8)} + \frac{1}{(2\text{x}3\text{x}8\text{x}6)} + \frac{1}{(2\text{x}3\text{x}8\text{x}6\text{x}2)} + \cdots .
\end{aligned}$$

Thus the modified Engel expansion of $\log(2)$ begins $[2, 3, 8, 6, 2, ...]$.

So far we have only considered the modified Engel expansion for real x in the interval $(0, 1]$. By analogy with what happens in the case of the Engel expansion of $x > 1$ we define the modified Engel expansion of a real $x > 1$ (x not an integer) with, say, $\text{floor}(x) = m$, to be equal to the modified Engel expansion of the fractional part of x prepended by a sequence of m 1's.

4. The modified Engel expansion of the quadratic irrational $(p-1)\sqrt{p^2-1}-(p^2-p-1)$

Let $p \geq 3$ be an integer and define the quadratic irrational

$$L(p) := \frac{1}{2}\left(p - \sqrt{p^2-4}\right).$$

The Engel expansion $[e_1, e_2, e_3, ...]$ of $L(p)$ was found by Lucas [Lu, Chapitre 18] and is given by $e_1 = p$ and $e_{n+1} = e_n^2 - 2$ for integer $n \geq 1$. This result is most easily obtained by iterating the identity

$$L(p) = \frac{1}{p} + \frac{L(p^2-2)}{p} \tag{16}$$

to obtain a representation for $L(p)$ as a series of Egyptian fractions

$$L(p) = \frac{1}{p} + \frac{1}{p(p^2-2)} + \frac{1}{p(p^2-2)(p^4-4p^2+2)} + \cdots$$

and then invoking the uniqueness of the Engel expansion.
In this section we shall prove a similar result for the modified Engel expansion of the quadratic irrational

$$Q(p) := (p-1)\sqrt{p^2-1} - (p^2-p-1),$$

where now $p \geq 2$.
Taking the place of (16) is the easily verified identity

$$Q(p) = \frac{1}{2} + \frac{1}{2(p+1)} + \frac{1}{2(p+1)(2p)} + \frac{Q(2p^2-1)}{2(p+1)(2p)}. \tag{17}$$

Let $[E_1, E_2, E_3, ...]$ denote the modified Engel expansion of $Q(p)$, that is, let

$$\begin{aligned} E_1 &= 1 + \mathrm{floor}\left(\frac{1}{Q(p)}\right) \\ E_n &= \mathrm{floor}\left(\frac{1}{h^{(n-2)}(Q(p))}\right)\left\{1 + \mathrm{floor}\left(\frac{1}{h^{(n-1)}(Q(p))}\right)\right\} \quad \text{for } n \geq 2. \end{aligned}$$

We know from Theorem 3.1(b) with $n = 3$ that

$$Q(p) = \frac{1}{E_1} + \frac{1}{E_1E_2} + \frac{1}{E_1E_2E_3} + \frac{1}{E_1E_2E_3}\frac{h^{(3)}(Q(p))}{\mathrm{floor}\left(\frac{1}{h^{(2)}(x)}\right)}. \tag{18}$$

We will show that the expansions (17) and (18) are identical. This is a consequence of the following result.

Lemma 4.1. Let $p \geq 2$ be an integer and let $Q(p) = (p-1)\sqrt{p^2-1} - (p^2-p-1)$. Then

(a) $\operatorname{floor}\left(\frac{1}{Q(p)}\right) = 1$

(b) $\operatorname{floor}\left(\frac{1}{h(Q(p))}\right) = p$

(c) $\operatorname{floor}\left(\frac{1}{h^{(2)}(Q(p))}\right) = 1.$

We assume, for the moment, the truth of Lemma 4.1, and show how the modified Engel expansion of $Q(p)$ follows from this.

It follows then from Lemma 4.1 that the modified Engel expansion for $Q(p)$ begins

$$E_1 = 2,\ E_2 = p+1,\ E_3 = 2p.$$

Inserting these values into (18) we get

$$Q(p) = \frac{1}{2} + \frac{1}{2(p+1)} + \frac{1}{2(p+1)(2p)} + \frac{h^{(3)}(Q(p))}{2(p+1)(2p)}.$$

Comparing this result with (17) shows that

$$Q(2p^2-1) = h^{(3)}(Q(p)).$$

Let us write $p\prime$ in place of $2p^2-1$, so that the previous result becomes

$$Q(p') = h^{(3)}(Q(p)). \tag{19}$$

Applying now lemma 4.1 to (19) gives

$$\begin{aligned}
\operatorname{floor}\left(\frac{1}{h^{(3)}\left(Q(p)\right)}\right) &= \operatorname{floor}\left(\frac{1}{Q(p')}\right) = 1 \\
\operatorname{floor}\left(\frac{1}{h^{(4)}\left(Q(p)\right)}\right) &= \operatorname{floor}\left(\frac{1}{h(Q(p'))}\right) = p' \\
\operatorname{floor}\left(\frac{1}{h^{(5)}(Q(p))}\right) &= \operatorname{floor}\left(\frac{1}{h^{(2)}(Q(p'))}\right) = 1
\end{aligned}$$

Thus the next three values in the modified Engel expansion of $Q(p)$ are

$$E_4 = 2,\ E_5 = p'+1,\ E_6 = 2p'.$$

We also have from (19)

$$h^{(6)}(Q(p)) = h^{(3)}(Q(p^{'})) = Q(p''), \quad \text{where } p'' = 2(p\prime)^2 - 1.$$

Clearly, we can continue in this manner and arrive at the following result.

Theorem 4.1. *The modified Engel expansion of the quadratic irrational* $Q(p) := (p-1)\sqrt{p^2-1} - (p^2-p-1)$ *,* $p \geq 2$*, has the form* $[2, a_1, a_2, 2, a_3, a_4, ..., 2, a_n, a_{n+1}, ...]$, where $a_1 = p$, $a_{2n} = a_{2n-1} + 1$ and $a_{2n+1} = 2a_{2n-1}^2 - 1$ for $n \geq 1$.

In order to complete the proof of this theorem it remains to prove Lemma 4.1. We require the following elementary inequalities, which are easily proved by considering the Taylor series expansions of $\sqrt{1-x}$ and $1/\sqrt{1-x}$ around $x = 0$.

For $p > 1$ there holds the inequalities

$$\sqrt{1 - \frac{1}{p^2}} < 1 - \frac{1}{2p^2} \tag{20}$$

$$\frac{1}{\sqrt{1 - \frac{1}{p^2}}} > 1 + \frac{1}{2p^2} \tag{21}$$

$$\frac{1}{\sqrt{1 - \frac{1}{p^2}}} > 1 + \frac{1}{2p^2} + \frac{3}{8p^4} + \frac{5}{16p^6} \tag{22}$$

Proof of Lemma 4.1

(a) $\mathrm{floor}\left(\frac{1}{Q(p)}\right) = 1$.

We have

$$\frac{1}{1 - Q(p)} = \frac{1}{(p-1)\left(p - \sqrt{p^2-1}\right)} = \frac{p + \sqrt{p^2-1}}{p-1} > 2 \quad \text{when } p \geq 2.$$

It follows that

$$\frac{1}{2} < Q(p) \leq 1$$

and consequently

$$\mathrm{floor}\left(\frac{1}{Q(p)}\right) = 1.$$

(b) $\mathrm{floor}\left(\frac{1}{h(Q(p))}\right) = p$.

From the definition (1) of the map $h(x)$ we have

$$\begin{aligned} h(Q(p)) &= \mathrm{floor}\left(\frac{1}{Q(p)}\right)\left(Q(p)\left(1 + \mathrm{floor}\left(\frac{1}{Q(p)}\right)\right) - 1\right) \\ &= 2Q(p) - 1 \quad \text{by (a).} \end{aligned} \tag{23}$$

Thus part (b) will be proved if we can show

$$\frac{1}{p+1} < 2Q(p) - 1 < \frac{1}{p}. \tag{24}$$

The left-hand inequality in (24) holds since

$$\begin{aligned}
2Q(p) - 1 &= 2(p-1)\sqrt{p^2-1} - 2p^2 + 2p + 1 \\
&= 2(p-1)\frac{p^2-1}{\sqrt{p^2-1}} - 2p^2 + 2p + 1 \\
&= 2(p-1)\frac{(p^2-1)}{p}\frac{1}{\sqrt{1-\frac{1}{p^2}}} - 2p^2 + 2p + 1 \\
&> 2(p-1)\frac{(p^2-1)}{p}\left(1 + \frac{1}{2p^2}\right) - 2p^2 + 2p + 1 \quad \text{by (21)} \\
&= \frac{1}{p} - \frac{1}{p^2} + \frac{1}{p^3} \\
&= \frac{1}{p+1} + \frac{1}{p^3(p+1)} \\
&> \frac{1}{p+1}.
\end{aligned}$$

Similarly, the right hand inequality in (24) follows from

$$\begin{aligned}
2Q(p) - 1 &= 2(p-1)\sqrt{p^2-1} - 2p^2 + 2p + 1 \\
&= 2(p-1)p\sqrt{1-\frac{1}{p^2}} - 2p^2 + 2p + 1 \\
&< 2(p-1)p\left(1 - \frac{1}{2p^2}\right) - 2p^2 + 2p + 1 \quad \text{by (20)} \\
&= \frac{1}{p}.
\end{aligned}$$

(c) $\mathrm{floor}\left(\frac{1}{h^{(2)}(Q(p))}\right) = 1.$

We have

$$\begin{aligned} h^2(Q(p)) &= h(h(Q(p)) \\ &= \mathrm{floor}\left(\frac{1}{h(Q(p))}\right)\left(h(Q(p))\left(1+\mathrm{floor}\left(\frac{1}{h(Q(p))}\right)\right)-1\right) \\ &= p\left\{(2Q(p)-1)(p+1)-1\right\} \quad \text{using (23) and (b).} \end{aligned}$$

Thus to prove (c) it will be sufficient to establish the inequalities

$$\frac{1}{2} < p\left\{(2Q(p)-1)(p+1)-1\right\} < 1. \tag{25}$$

The left-hand inequality in (25) holds since

$$\begin{aligned} p\left\{(2Q(p)-1)(p+1)-1\right\} &= p\left\{2(p^2-1)\sqrt{p^2-1}-2p^3+3p\right\} \\ &= p\left\{2(p^2-1)^2\frac{1}{\sqrt{p^2-1}}-2p^3+3p\right\} \\ &= p\left\{\frac{2(p^2-1)^2}{p}\frac{1}{\sqrt{1-\frac{1}{p^2}}}-2p^3+3p\right\} \\ &> p\left\{\frac{2(p^2-1)^2}{p}(1+\frac{1}{2p^2}+\frac{3}{8p^4}+\frac{5}{16p^6})-2p^3+3p\right\} \quad \text{by (22)} \\ &= \frac{3}{4}+\frac{p^2(p^2-4)+5}{8p^6} \\ &> \frac{1}{2} \quad \text{since } p \geq 2. \end{aligned}$$

The right-hand inequality in (25) follows from

$$\begin{aligned} p\left\{(2Q(p)-1)(p+1)-1\right\} &= p\left\{2(p^2-1)\sqrt{p^2-1}-2p^3+3p\right\} \\ &= p\left\{2(p^2-1)p\sqrt{1-\frac{1}{p^2}}-2p^3+3p\right\} \\ &< p\left\{2(p^2-1)p\left(1-\frac{1}{2p^2}\right)-2p^3+3p\right\} \quad \text{by (20)} \\ &= 1. \end{aligned}$$

Thus Lemma 4.1 has been proved and with it the proof of Theorem 4.1 has been completed.

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