ESTIMATE OF THE SUM OVER INVERSE PRODUCTS OF ADJACENT PRIME PAIRS

RICHARD J. MATHAR

Abstract. This is an overview of a recent computation of sequence A209329.

1. Introduction

1.1. **Motivation.** Sums over powers of products of inverse adjacent primes occur in a geometrical construction of circumscribed regular polygons with prime numbers of edges [1, App. B].

1.2. **Numerics.** If the next prime after the odd prime p is replaced by its lower estimate $p+2$ we arrive at [4, A185380]

(1.1)
$$H(1,2) \equiv \sum_p \frac{1}{p(p+2)} = \frac{1}{2\times 4} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \frac{1}{7\times 9} + \frac{1}{11\times 13} + \frac{1}{13\times 15} + \cdots$$
$$\approx 0.26367206176115317874984218823377675308749631839675680212223 8.$$

This class of infinite sums over inverse polynomials over primes is evaluated numerically by accumulating the first terms up to some convenient limit $p \le M$, then writing the infinite remainder as a logarithm of the exponential, where the exponential of the sum turns into a product over the primes $p > M$ of exponentials. The exponential of the inverse term is then written as a Taylor series in $1/p$, and this converted by an Euler transform into an infinite product of powers of the Riemann zeta function. The generic idea is equivalent to the standards of evaluating the Prime Zeta Function.

In the same spirit, the prime before the higher prime in the product could be replaced by an upper estimate, building the infinite sum

(1.2)
$$H(1,-2) \equiv \sum_{\mathrm{odd}\, p} \frac{1}{p(p-2)} = \frac{1}{3\times 1} + \frac{1}{5\times 3} + \frac{1}{7\times 5} + \frac{1}{11\times 9} + \frac{1}{13\times 11} + \cdots$$
$$\approx 0.46354235297066369361460556394340891112783722087116318.$$

Since the prime gaps of the odd primes are always 2 or larger, the two modified sequences and their rather accurately known series limits establish rigid upper and

Date: February 10, 2013.
2010 *Mathematics Subject Classification.* Primary 33B20, 28-04; Secondary 65D20.
Key words and phrases. Brun's constant, twin primes, Euler products.

lower bounds for the remainder of the irregular partial sums of the adjacent odd primes:

$$\begin{aligned}(1.3)\quad &\frac{1}{3\times 5}+\frac{1}{5\times 7}+\frac{1}{9\times 11}+\frac{1}{11\times 13}+\frac{1}{15\times 17}+\cdots\\ &< \frac{1}{3\times 5}+\frac{1}{5\times 7}+\frac{1}{7\times 11}+\frac{1}{11\times 13}+\frac{1}{13\times 17}+\cdots\\ &< \frac{1}{3\times 5}+\frac{1}{5\times 7}+\frac{1}{7\times 9}+\frac{1}{11\times 13}+\frac{1}{13\times 15}+\cdots .\end{aligned}$$

The first sum equals $H(1,-2)-1/3 \approx 0.1302090$ and the the third sum equals $H(1,2)-1/8 \approx 0.138672061761$. An equivalent chain of inequalities is correct for any sum of remainders of the three expressions are summed to the same number of terms. Because the remainders of the first and third sum are then known to high precision, a strictly bracketed estimate of the second sum is established. In summary [4, A209329],

$$\sum_{\text{odd } p_j} \frac{1}{p_j p_{j+1}} \approx 0.134426509691733228215301 \pm 4.2\times 10^{-21}. \tag{1.4}$$

1.3. **Higher Powers.** The higher powers of these sums are more accessible because convergence is a smaller problem then.

Definition 1. *(Hurwitz type Prime Zeta Function)*

$$H(s,a) \equiv \sum_{p>-a} \frac{1}{p^s (p+a)^s}. \tag{1.5}$$

For the squares, $s=2$, the technique of the previous section yields

$$(1.6)\quad H(2,2) = \sum_p \frac{1}{[p(p+2)]^2}$$
$$\approx 0.02123578646222644858836678157934276949777665756430185468578263219207687012616$$

$$(1.7)\quad H(2,-2) = \sum_{p>2} \frac{1}{[p(p-2)]^2}$$
$$\approx 0.1165570525201333269890282585614520314626290638474605867931965642064.$$

(1.8)
$$\sum_{\text{odd } p_j} \frac{1}{(p_j p_{j+1})^2} \approx 0.0055195227745598147590742993699822088427 \pm 3.7\times 10^{-37}.$$

For the cubes, $s=3$, we obtain

$$(1.9)\quad H(3,2) = \sum_p \frac{1}{[p(p+2)]^3}$$
$$\approx 0.00227727483741173573231058892139695549334781832254059074553350270055934520077;$$

$$(1.10)\quad H(3,-2) = \sum_{p>2} \frac{1}{[p(p-2)]^3}$$
$$\approx 0.037358132985391123730725252009030989930376660504802282336860779677358460829690;$$

(1.11)
$$\sum_{\text{odd}\, p_j} \frac{1}{(p_j p_{j+1})^3} \approx 0.000322292776977036899122902204863290099228085021977585222 \pm 1.7\times 10^{-54}.$$

2. Splitting Brun's Constant

2.1. **Refinement to Prime Constellations.** The value of (1.4) is a sum over all possible even prime gaps of first powers of primes in the denominator: $T_2(1) + T_4(1) + T_6(1) + T_8(1) + \cdots$ This previous section showed that a refinement of the sum to specific classes of prime gaps may have some computational advantages.

Definition 2. *(Prime gaps.) We define the j-th prime gap as the distance of the j-th prime to the next prime:*

$$g_j \equiv p_{j+1} - p_j. \tag{2.1}$$

We consider the following family of constants:

Definition 3.

$$T_g(s) \equiv \sum_{\text{odd}\, p_j : g_j = d} \frac{1}{[p_j(p_j+d)]^s}. \tag{2.2}$$

Each term in these sums might be decomposed into partial fractions because it is an inverse product of a number p by a number at some distance d. The power $s = 1$ is

$$\frac{1}{p(p+d)} = \frac{1}{d}\left(\frac{1}{p} - \frac{1}{p+d}\right), \tag{2.3}$$

and $s = 4$ is
(2.4)
$$\frac{1}{p^4(p+d)^4} = -\frac{20}{d^7}\left(\frac{1}{p} - \frac{1}{p+d}\right) + \frac{10}{d^6}\left(\frac{1}{p^2} + \frac{1}{(p+d)^2}\right) - \frac{4}{d^5}\left(\frac{1}{p^3} - \frac{1}{(p+d)^3}\right) + \frac{1}{d^4}\left(\frac{1}{p^4} + \frac{1}{(p+d)^4}\right).$$

For general integer n [4, A092392]

$$\frac{1}{p^n(p+d)^n} = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} (-1)^{n-k} \frac{1}{d^{2n-k}} \left(\frac{1}{p^k} + \frac{(-)^k}{(p+d)^k}\right). \tag{2.5}$$

Remark 1. *This may be written as*

$$\frac{(-d^2)^n}{p^n(p+d)^n} = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} (-d)^k \left(\frac{1}{p^k} + \frac{(-)^k}{(p+d)^k}\right). \tag{2.6}$$

The matrix inverse of this equation is [4, A117362][3, §4.3]

$$\frac{d^n}{n}\left[\frac{1}{p^n} + \frac{(-)^n}{(p+d)^n}\right] = \sum_{k=1}^{n} \frac{1}{k} \binom{k}{n-k} \frac{d^{2k}}{p^k(p+d)^k}. \tag{2.7}$$

So instead of calculating (2.2) directly one might substitute the right hand side with (2.5) and build a library of "lower" and "upper" constants targeting the k-th power of lower and upper primes with gaps of size d:

Table 1. The partial sums of the L and U sum for a gap 2 of adjacent primes if summed up to p_j. See also [4, A209328].

j	$\frac{1}{2}L_2(1)$	$\frac{1}{2}U_2(1)$	$\frac{L_2(1)-U_2(1)}{2}$
400100000	0.500692690603974682267	0.392708715653053448393	0.10798397495092123387
800600000	0.501579877767927487341	0.39359590281686138007	0.107983974951066107265
160940000	0.502418023345598624878	0.394434048394466436314	0.107983974951132188563

Table 2. The partial sums of the L and U sum for a gap 4 of adjacent primes if summed up to p_j.

j	$L_4(1)$	$U_4(1)$	$L_4(1)-U_4(1)$
450000000	0.584898527909973887118	0.497406096444372827096	0.08749243146560106
900000000	0.586652264332792268292	0.499159832866684671845	0.08749243146610759
1800000000	0.588299650478246198600	0.500807219011908238893	0.08749243146633795

Table 3. The partial sums of the L and U sum for a gap 6 of adjacent primes if summed up to p_j.

j	$L_6(1)$	$U_6(1)$	$L_6(1)-U_6(1)$
450000000	0.470726598296217343800	0.446422523142666394791	0.0243040751535509
900000000	0.473909200210918614083	0.449605125055988981447	0.0243040751549296
1800000000	0.476907825029244235122	0.452603749873685749222	0.0243040751555584

Definition 4. *(Lower and Upper d-prime Zeta Functions)*

$$L_d(s) \equiv \sum_{p_j: g_j = d} \frac{1}{p_j^s}; \tag{2.8}$$

$$U_d(s) \equiv \sum_{p_j: g_j = d} \frac{1}{(p_j + d)^s}. \tag{2.9}$$

2.2. **Prime Constellation** $(p, p+2)$**.** Brun's constant is the sum over inverse twin primes [4, A065421],

$$B_2 \equiv \sum_{p_j, g_j=2} \left(\frac{1}{p_j} + \frac{1}{p_j+2}\right) = \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots$$
$$= L_2(1) + U_2(1) \approx 1.902160583104, \tag{2.10}$$

where the final three digits may be inaccurate. The $L_d(1)+U_d(1)$ are converging very slowly, whereas the differences are known at least to 11 digits (Table 1).

Combining the estimate $L_2(1)-U_2(1) \approx 0.107983974951$ with the value of (2.10) we obtain $L_2(1) \approx 1.05906426650$ and $U_2(1) \approx 0.84309631660$.

2.3. **Prime Constellations** $(p, p+4)$**,** $(p, p+6)$ **and** $(p, p+8)$**.** For cousin primes and primes with a gap of 6, the constants B_4 and B_6 have also been estimated [4, A194098][2]. The associated convergence of $L_d(1)-U_d(1)$ is reported in tables 2–4.

TABLE 4. The partial sums of the L and U sum for a gap 8 of adjacent primes if summed up to p_j.

j	$L_8(1)$	$U_8(1)$	$L_8(1) - U_8(1)$
450000000	0.119518969489285100415	0.118137436265303315256	0.0013815332239817851
900000000	0.120958287195567593747	0.119576753970754683536	0.0013815332248129102
1800000000	0.122318691819565715309	0.120937158594372469411	0.0013815332251932458

Summarizing these four tables and (2.3), the total contribution of gaps 2, 4, 6 and 8 to (1.4) is at least

(2.11)

$$0.107983974951132188563+\frac{1}{4}\times 0.08749243146633795+\frac{1}{6}\times 0.0243040751555584 + \frac{1}{8}\times 0.0013815332251932458 \approx 0.134080453663458.$$

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URL: `http://www.mpia.de/~mathar`
E-mail address: `mathar@mpia.de`

R. J. MATHAR, HOESCHSTR. 7, 52372 KREUZAU, GERMANY