PROOF OF CONJECTURE IN A209260

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The purpose of this note is to prove a conjecture stated by the author in Sloane's sequence A209260 [5] in *The On-Line Encyclopedia of Integer Sequences* [2]. The conjecture is restated below as Proposition 1, after some preliminaries.

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers. For $r \in \mathbb{N} \cup \{0\}$, let $T(r)$ denote the r-th triangular number defined by $T(r) = \frac{r(r+1)}{2}$. The triangular numbers are the sequence $\{0, 1, 3, 6, 10, 15, \ldots\}$ (Sloane's A000217 [3]).

Definition 1. *Let $n \in \mathbb{N}$, and let S_n be the set*

$$\begin{aligned} S_n &= \{\text{union of the first } n \text{ rows of A209260}\} \\ &= \bigcup_{\substack{m \in \{1,\ldots,n\} \\ k \in \{1,\ldots,m\}}} \text{A141419}(m, k) \end{aligned}$$

of all entries in the first n rows of Sloane's sequence A141419 [4]*, with the entry in row m and column k defined by* $\text{A141419}(m, k) = T(m) - T(m - k)$.

The following lemma, conjectured by the author, is stated here separately since proof is based on one given by Charles R. Greathouse IV in [1].

Lemma 1. *Let $n \in \mathbb{N}$. If $2n + 1$ is composite, and d and h are positive integers such that $2n + 1 = dh$, with $1 < d < 2n + 1$, then $d + \frac{h-1}{2} \leq n$.*

Proof. Suppose that $2n + 1$ is composite, and d and h are positive integers such that $2n + 1 = dh$, with $1 < d < 2n + 1$. Clearly $\frac{h-1}{2} \leq \frac{n-1}{3}$, since $d \geq 3$. Therefore $d + \frac{h-1}{2} \leq \sqrt{2n + 1} + \frac{n-1}{3} \leq n$, as required. □

Proposition 1. *Let $n \in \mathbb{N}$. A necessary and sufficient condition for $2n + 1 \in S_n$ is that $2n + 1$ be composite.*

Proof. Suppose that $2n+1 \in S_n$. Then $2n+1 = T(m)-T(m-k)$, by the definition of S_n, or what is the same thing,

$$2n + 1 = k\left(m - \frac{k-1}{2}\right), \tag{1}$$

for some $m \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$. If $k = 1$, then $2n + 1 = m < 2n + 1$, a contradiction. Similarly, if $k = 2$, then $2n + 1 = 2m - 1 < 2n + 1$, a contradiction. Hence $k \geq 3$, and we note that k cannot be a power of 2, since $2n + 1$ is odd. Now in (1), if k is odd, then $m - \frac{k-1}{2}$ is a positive integer and necessarily odd. On the other hand, if k is even, then $k = 2^u(2v + 1)$, say, for some $u, v \in \mathbb{N}$. However, only $u = 1$ is possible since, for $u \geq 2$, the right side of (1) would be even, contradicting the fact that the left side is odd. Therefore, if k is even, then $k = 2\,(2v + 1)$, so that $2n + 1 = (2v + 1)\,(2m - k + 1)$ in which $2m - k + 1$ is positive and necessarily odd.

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In either case, k even or odd, it follows that $2n+1$ must be composite.[1] Therefore it is a necessary condition that $2n+1$ be composite.

Conversely, suppose that $2n+1$ is composite. Then there are positive integers d and h such that $3 \le d \le h < 2n+1$, d divides $2n+1$, and $2n+1 = dh$. We have that $1 < h - \frac{d-1}{2} < h + \frac{d-1}{2}$, and $2n+1$ has representation as a sum of d consecutive positive integers of the form

$$2n+1 = dh = \sum_{j=-\frac{d-1}{2}}^{\frac{d-1}{2}} (h+j)\,,$$

or equivalently, letting $M = h + \frac{d-1}{2}$,

$$\begin{aligned}
2n+1 &= \sum_{a=0}^{d-1} (M-a) \\
&= M + (M-1) + \cdots + (M-d+1) \\
&= Md - \frac{d(d-1)}{2} \\
&= \frac{M(M+1)}{2} - \frac{M(M+1)}{2} + Md - \frac{d(d-1)}{2} \\
&= \frac{M(M+1)}{2} - \frac{M(M+1) - 2Md + d(d-1)}{2} \\
&= \frac{M(M+1)}{2} - \frac{(M-d)(M-d+1)}{2} \\
&= T(M) - T(M-d) \\
&= \text{A141419}(M,d).
\end{aligned}$$

Since $M = h + \frac{d-1}{2} \le n$, by Lemma 1, we conclude that $2n+1 \in S_n$. Therefore it is a sufficient condition that $2n+1$ be composite. This completes the proof. □

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References

[1] C. R. Greathouse IV, Personal communications, 2013.
[2] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `https://oeis.org`.
[3] N. J. A. Sloane, `https://oeis.org/A000217`.
[4] N. J. A. Sloane, `https://oeis.org/A141419`.
[5] N. J. A. Sloane, `https://oeis.org/A209260`.

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[1] Alternatively, since $k \notin \{1,2\}$ in the above argument, we note simply that $2n+1$ cannot be a prime and $\in S_n$ simultaneously and so must be composite.