Suppose that

$$M_n = (2\min\{i,j\}-1)_{1\le i,j\le n} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 3 & \cdots & 3 \\ 1 & 3 & 5 & \cdots & 5 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 3 & 5 & \cdots & 2n-1 \end{pmatrix},$$

$T_n = \det(xI_n - M_n)$ be the characteristic polynomial of M_n. We have

$$\begin{aligned}
T_n &= (-1)^n \det \begin{pmatrix} 1-x & 1 & 1 & \cdots & 1 & 1 \\ 1 & 3-x & 3 & \cdots & 3 & 3 \\ 1 & 3 & 5-x & \cdots & 5 & 5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 3 & 5 & \cdots & 2n-3-x & 2n-3 \\ 1 & 3 & 5 & \cdots & 2n-3 & 2n-1-x \end{pmatrix} \\
&= (-1)^n \det \begin{pmatrix} 1-x & 1 & 1 & \cdots & 1 & 1 \\ 1 & 3-x & 3 & \cdots & 3 & 3 \\ 1 & 3 & 5-x & \cdots & 5 & 5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 3 & 5 & \cdots & 2n-3-x & 2n-3 \\ 0 & 0 & 0 & \cdots & x & 2-x \end{pmatrix} \\
&= -x(-1)^n \det \begin{pmatrix} 1-x & 1 & 1 & \cdots & 1 & 1 \\ 1 & 3-x & 3 & \cdots & 3 & 3 \\ 1 & 3 & 5-x & \cdots & 5 & 5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 3 & 5 & \cdots & 2n-5-x & 2n-5 \\ 1 & 3 & 5 & \cdots & 2n-5 & 2n-3 \end{pmatrix} + (x-2)T_{n-1} \\
&= -x(-1)^n \left(\det \begin{pmatrix} 1-x & 1 & 1 & \cdots & 1 & 1 \\ 1 & 3-x & 3 & \cdots & 3 & 3 \\ 1 & 3 & 5-x & \cdots & 5 & 5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 3 & 5 & \cdots & 2n-5-x & 2n-5 \\ 1 & 3 & 5 & \cdots & 2n-5 & 2n-3-x \end{pmatrix} \right. \\
&\qquad \left. + \det \begin{pmatrix} 1-x & 1 & 1 & \cdots & 1 & 0 \\ 1 & 3-x & 3 & \cdots & 3 & 0 \\ 1 & 3 & 5-x & \cdots & 5 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 3 & 5 & \cdots & 2n-5-x & 0 \\ 1 & 3 & 5 & \cdots & 2n-5 & x \end{pmatrix} \right) + (x-2)T_{n-1} \\
&= xT_{n-1} - x^2T_{n-2} + (x-2)T_{n-1} = (2x-2)T_{n-1} - x^2T_{n-2}, \quad n \ge 2,
\end{aligned}$$

with $T_0 = 1$, $T_1 = x - 1$, so

$$T_n = \frac{(x - 1 + \mathrm{i}\sqrt{2x-1})^n + (x - 1 + \mathrm{i}\sqrt{2x-1})}{2},$$

and

$$T_n\left(\frac{1}{1+\cos\theta}\right) = \frac{(-1)^n((\cos\theta - \mathrm{i}\sin\theta)^n + (\cos\theta + \mathrm{i}\sin\theta)^n)}{2(1+\cos\theta)^n} = \frac{(-1)^n \cos n\theta}{(1+\cos\theta)^n}, \quad \forall\theta \in [0, \pi),$$

from which we deduce that

$$T_n\left(\frac{1}{1+\cos\dfrac{(2k-1)\pi}{2n}}\right) = 0, \quad k = 1, \cdots, n.$$

Since T_n is a polynomial of degree n as the characteristic polynomial of an $n \times n$ matrix, we conclude that the n roots of T_n are $\dfrac{1}{1+\cos\dfrac{(2k-1)\pi}{2n}}$, $k = 1, \cdots, n$.