Derivation of explicit formulas

\( a(n) \): Number of ways of making change for \( n \) cents using coins of \( 1 = c(1) < c(2) < \ldots < c(k) \) cents.

General recurrence (explanation in \( \text{A187243} \)):

\[
\begin{align*}
\text{f}(n, 1) &= 1 \\
\text{1} < j \leq \text{k}: \quad \text{f}(0, j) &= 1 \text{ or, for } n > 0: \\
&= \sum_{m=0}^{n \div c(j)} \text{f}(n - m \cdot c(j), j - 1) \\
&= \sum_{m=0}^{n \mod c(j) + m \cdot c(j), j - 1} \text{f}(n \mod c(j) + m \cdot c(j), j - 1) \text{, transformation } m \rightarrow n \div c(j) - m \\
a(n) &= \text{f}(n, k)
\end{align*}
\]

Integer division: \( a \div b = (a - a \mod b)/b = \text{floor}(a/b) \).

\[\text{A187243} \], \( c(1) = 1, c(2) = 5, c(3) = 10 \):

\[ a(n) = (q+1) \cdot (q+1+s) \text{ with } q = n \div 10 \text{ and } s = (n \mod 10) \div 5 \]

Derivation:

Let \( q = n \div 10 \) and \( r = n \mod 10 \), i.e. \( n = 10q + r \) with \( 0 \leq r \leq 9 \).

\[
\begin{align*}
\text{f}(n, 1) &= \sum_{m=0}^{r \div 10} \text{f}(r + 10m, 2), \text{ see recurrence (b)} \\
&= \sum_{m=0}^{r \div 2} \text{f}(r' + 5 \cdot (2m + s), 2) \text{ with } s = r \div 5 \text{ and } r' = r \mod 5 \\
\text{f}(5 \cdot (r' + 2m + s), 2) &= \sum_{m=0}^{r \div 2} \text{f}(r' + 5 \div 2 (2m + s), 1) = \sum_{m=0}^{r \div 2} (2m + 1 + 5 \cdot s, 1) = 2m + 1 + s \\
\text{a}(n) &= \text{f}(n, 3) = \sum_{m=0}^{r \div 2} (2m + 1 + s) = q(q+1) + (q+1)(1+s) = (q+1)(q+1+s), \text{ q.e.d.}
\end{align*}
\]

Another formula:

\[ a(n) = \text{f}(n, 3) = \text{floor} \left( \frac{(x+10)^2}{100} \right) \text{ with } x = n - n \mod 5 \]

\[ \text{A001299} \], \( c(1) = 1, c(2) = 5, c(3) = 10, c(4) = 25 \):

\[ a(n) = \text{round} \left( ((100x^3 + 135x^2 + 53x)/6) + 1 \right) \text{ with } x = \text{floor}(n/5)/10 \]

Derivation:

1) \( n = 50x \) (note that \( 50 = \text{LCM}(5,10,25) \)):

\[
\begin{align*}
\text{f}(n, 3) &= (q+1) \cdot (q+1+s) \text{ with } q = n \div 10 \text{ and } s = (n \mod 10) \div 5 \\
\text{f}(n, 4) &= \sum_{m=0}^{n \div 25} \text{f}(25m, 3) \\
a) m=2\mu, f(50\mu) = (5\mu+1)^2 \\
b) m=2\mu+1, f(25+50\mu) = (5\mu+3)(5\mu+4) \\
f(n,4) &= \sum_{m=0}^{x} \sum_{\mu=0}^{n/50} f(50\mu, 3) \\
&= \sum_{\mu=0}^{x} (25\mu^2 + 10\mu + 1) + \sum_{\mu=0}^{x} (25\mu + 12) \\
&= \sum_{\mu=0}^{x} (50\mu^2 + 45\mu + 13) + 25x^2 + 10x + 1 \\
&= \frac{50}{6} \cdot x(x - 1)(2x - 1) + \frac{45}{2} \cdot x(x - 1) + 13x + 25x^2 + 10x + 1 \\
&= \frac{5}{6} \cdot (100x^3 + 135x^2 + 53x + 6) (1)
\end{align*}
\]

2) Other cases:

Obviously: \( a(n_1) = a(n_2) \) when \( n_1 \div 5 = n_2 \div 5 \) because GCD(5,10,25) = 5.
Or let \( n = 5y + z \) with \( z = n \mod 5 \). When the change for \( 5y \) cents is done the change for the remaining \( z \) cents is fixed.

As a special result, formula (1) holds for \( n = 50x + r \) with \( 0 \leq r \leq 4 \):

\[
a(n) = \frac{1}{6} (100x^3 + 135x^2 + 53x + 6) \quad \text{with } x = \floor{n/5}/10 \quad \text{instead of } x = n/50 \quad (2)
\]

3) Remaining cases:

\( n = 50x + 5y, \ 1 \leq y \leq 9 \)

The analysis of these cases is straightforward but arduous. The deviations from formula (2) are small and can be compensated by the round-function.

\[\textbf{A000008} \quad c(1) = 1, \ c(2) = 2, \ c(3) = 5, \ c(4) = 10.\]

\[a(n) = (q + 1) \left( \text{round} \left( \frac{(n+4)^2}{20} \right) - \frac{1}{6} q(3n - 10q + 7) \right)\]

\[f(n, 3) = \text{round} \left( \frac{(n+4)^2}{20} \right), \text{ see A000115}\]

\[f(n, 4) = \sum_{m=0}^{q} f(n - 10m, 3), \ q = n \div 10, \text{ see recurrence (a)}\]

\[f(n-10m, 3) = \text{round} \left( \frac{(n+4-10m)^2}{20} \right) = f(n, 3) - (n + 4)m + 5m^2\]

\[f(n, 4) = (q+1) \cdot f(n, 3) - \frac{n+4}{2} q(q + 1) + \frac{5}{6} q(q + 1)(2q + 1)\]

\[f(n, 4) = (q + 1) \left( \text{round} \left( \frac{(n+4)^2}{20} \right) - \frac{1}{6} q(3n - 10q + 7) \right)\]