Maple-assisted derivation of formula for A185554

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We begin by enumerating the possible configurations for a 2×4 sub-array, where each element is equal to at most two of its horizontal and vertical neighbours.

An array $\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{bmatrix}$ is encoded as the list $[b_1, b_2, ..., b_8]$.

.

```
> Configs:= NULL:
  for x from 3^8 to 2*3^8-1 do
    L:= convert(x,base,3)[1..8];
    if nops({L[1],L[2],L[3],L[6]}) > 1 and nops({L[2],L[3],L[4],L
  [7]}) > 1
      and nops({L[2],L[5],L[6],L[7]}) > 1 and nops({L[3],L[6],L[7],
  L[8]}) > 1
       then Configs:= Configs, L
    fi
  od:
  Configs:= [Configs]: nops(Configs);
```

$$5718 \tag{1}$$

There are 5718 possible configurations.

Consider the 5718×5718 transition matrix T such that $T_{ij} = 1$ if the bottom two rows of a 3×4 sub-array could be in configuration i while the top two rows are in configuration j (i.e. the middle row is compatible with both i and j, and each element in that row is equal to 1 or 2 of its horizontal and vertical neighbours), and 0 otherwise. The following Maple code computes it.

```
> q:= proc(i,j) local Ci, Cj;
    Ci:= Configs[i]; Cj:= Configs[j];
    if Ci[1..4] <> Cj[5..8] then return false fi;
    {numboccur(Ci[1],[Cj[1],Ci[2],Ci[5]]), numboccur(Ci[2],[Cj[2],
  Ci[1],Ci[3],Ci[6]]),
        numboccur(Ci[3],[Cj[3],Ci[2],Ci[4],Ci[7]]), numboccur(Ci
  [4],[Cj[4],Ci[3],Ci[8]])} subset {1,2}
  end proc:
  T:= Matrix(5718,5718, storage=sparse):
  for i from 1 to 5718 do for j from 1 to 5718 do
    if q(i,j) then T[i,j]:= 1 fi od od:
```

Since the constraints are symmetric under permutations of 0,1,2, $a(n) = u\,T^{n-2}\,v$ for $n \geq 2$ where u and v are row and column vectors respectively with $u_i = 1$ for i corresponding to configurations that could be the bottom two rows of an array and have lower left entry 0, 0 otherwise (so each member of the bottom row is equal to 1 or 2 of its neighbours), and $v_i = 1$ for i corresponding to configurations that could be the top two rows, 0 otherwise. The following Maple code produces these vectors.

```
> uf:= proc(i) local Ci;
    Ci:= Configs[i];
    if Ci[5] = 0 and ((Ci[5]=Ci[1] or Ci[5]=Ci[6]) and member
  (numboccur(Ci[6],[Ci[5],Ci[2],Ci[7]]),{1,2}) and member(numboccur
  (Ci[7],[Ci[6],Ci[3],Ci[8]]),{1,2}) and (Ci[8]=Ci[7] or Ci[8]=Ci
  [4])) then 1 else 0 fi
  end proc:
  u:= Vector[row](5718,uf):
  vf:= proc(i) local Ci;
    Ci:= Configs[i];
    if (Ci[1]=Ci[5] or Ci[1]=Ci[2]) and member(numboccur(Ci[2],[Ci
  [1],Ci[6],Ci[3]]),{1,2}) and member(numboccur(Ci[3],[Ci[2],Ci[7],
  Ci[4]]),{1,2}) and (Ci[4]=Ci[3] or Ci[4]=Ci[8]) then 1 else 0 fi
  end proc:
  v:= Vector[column](5718,vf):
```

To check, here are the first few entries of our sequence. For future use, we precompute values of $T^n v$.

```
> Tnv[0]:= v:
  for n from 1 to 120 do Tnv[n]:= T . Tnv[n-1] od:
  seq(u . Tnv[n], n=0..20);
```

120, 2530, 58458, 1317540, 30132444, 684657326, 15594228288, 354894362166, 8078976180614, 183894888689408, 4185999926427220, 95284617951103300, 2168945067666040200, 49371181800142293644, 1123825227937533564738, 25581377597284008562428, 582303119481233722471478, 13254833954779407415752780, 301716785086455011043777576, 6867910857293036216335063986, 156332699887152127792146508580 **(2)**

We form the matrix with columns $T^n v$, $n = 0 \ldots 120$ and check its rank.

```
> M:= Matrix([seq(Tnv[i],i=0..120)]):
```

```
> LinearAlgebra:-Rank(M);
```

101 **(3)**

The rank turns out to be 101. This suggests that $T^n v$ for $n = 0 .. 101$ are linearly dependent. We find a nonzero vector w such that $\sum_{n=0}^{101} w_n T^n v = 0$.

```
> w:= op(1,LinearAlgebra:-NullSpace(M[.., 1..102])):
```

We thus have the linear recurrence $\sum_{i=0}^{101} w_i a(n+i) = u \sum_{i=0}^{101} w_i T^{n+i-2} v = 0$ for $n \geq 2$. It turns out to be also true for $n = 1$.

```
> n:= 'n': rec:= sort(add(w[i+1]*a(n+i),i=0..101),[seq(a(n+i),i=0.
  .101)])=0;
```

$rec := 29514790517935282585 6\,a(n) - 33204139332677192908 8\,a(n+1)$ **(4)**

$- 19415198137579303075 84\,a(n+2) - 20447062884202431119 36\,a(n+3)$

$- 12163321873602235596 8\,a(n+4) + 81823199469918119526 40\,a(n+5)$

$+ 25463154134769180082 176\,a(n+6) + 40449176880620098289 664\,a(n+7)$

$+ 40091104118497171472 384\,a(n+8) + 27903408341499141685 248\,a(n+9)$

$- 69419627796537991495 68\,a(n+10) - 52598494670443536973 824\,a(n+11)$

$- 53854491431912523956 224\,a(n+12) - 29886960998247227719 68\,a(n+13)$

$+ 50190706150366219599 872\,a(n+14) + 68475539393300238696 448\,a(n+15)$

$$
\begin{aligned}
&+ 969756883294031991275552\, a(n+16) + 303587399183248486563384\, a(n+17) \\
&- 2192895574625854485954456\, a(n+18) - 5396580928901912754913288\, a(n+19) \\
&- 9455594220662308250583304\, a(n+20) - 11600283584811146946805576\, a(n+21) \\
&- 10728245276225155712942088\, a(n+22) - 8003299697500426401218566\, a(n+23) \\
&- 3772737645750978714009600\, a(n+24) - 16329712415552121811763322\, a(n+25) \\
&- 10068859537765406500454444\, a(n+26) + 940100479183573127004166\, a(n+27) \\
&+ 3430281345070028963184644\, a(n+28) + 5413488521790257574707200\, a(n+29) \\
&+ 5440850889300725360230400\, a(n+30) + 3910321488694219641978888\, a(n+31) \\
&+ 2656075552624897273364488\, a(n+32) + 1507301449931038904811522\, a(n+33) \\
&+ 295240383659737187614722\, a(n+34) - 1136944960148525268336644\, a(n+35) \\
&- 2132194110593244880568322\, a(n+36) - 1886661921023516319088644\, a(n+37) \\
&- 999688868358105990103044\, a(n+38) - 182672685297005952225288\, a(n+39) \\
&+ 1208805340219368380211222\, a(n+40) + 1542094473287983324569666\, a(n+41) \\
&+ 2308339433056872104704000\, a(n+42) + 1875074055162278497190444\, a(n+43) \\
&+ 1031964228409363218137666\, a(n+44) - 32330139998658468211222\, a(n+45) \\
&- 11029870868762640496800\, a(n+46) + 23384911753103254930088\, a(n+47) \\
&+ 15401764729456974255566\, a(n+48) - 837008290049823360166\, a(n+49) \\
&- 17006131558078046520722\, a(n+50) - 7789295378108539183000\, a(n+51) \\
&- 11511107198339221223555\, a(n+52) + 422183249647589830022\, a(n+53) \\
&- 10016469039796030058\, a(n+54) - 9535006732908510434\, a(n+55) \\
&+ 23908385779591152069\, a(n+56) + 4029734507254683854\, a(n+57) \\
&+ 28551917408841809903\, a(n+58) + 13447047684495397700\, a(n+59) \\
&+ 217326932949056515000\, a(n+60) - 361416829163184945\, a(n+61) \\
&- 170811285891745950\, a(n+62) + 1309995955119981109\, a(n+63) \\
&- 266367690925611625\, a(n+64) + 160779258939467112\, a(n+65) \\
&- 397713205516343731\, a(n+66) + 141101005337997876\, a(n+67) \\
&- 97827116391337025\, a(n+68) + 68827142868696976\, a(n+69) \\
&- 37995917837230503\, a(n+70) + 16708334020463824\, a(n+71) \\
&- 8849643836131979\, a(n+72) + 3246879448312390\, a(n+73) \\
&- 1391389040465161\, a(n+74) + 1139770021269883\, a(n+75) \\
&- 690758571385125\, a(n+76) + 385093583785430\, a(n+77) - 205561890798248\, a(n \\
&+ 78) + 77740031051801\, a(n+79) - 20243817514119\, a(n+80) \\
&+ 4458240266632\, a(n+81) - 496296898921\, a(n+82) + 375565507264\, a(n+83) \\
&- 254308155324\, a(n+84) + 27484219476\, a(n+85) + 12000869775\, a(n+86) \\
&- 5211771316\, a(n+87) + 576808421\, a(n+88) + 485584116\, a(n+89) \\
&- 72333261\, a(n+90) - 56821843\, a(n+91) + 5854830\, a(n+92) + 4793870\, a(n \\
&+ 93) - 308510\, a(n+94) - 244487\, a(n+95) - 1397\, a(n+96) + 7766\, a(n+97) \\
&+ 885\, a(n+98) - 184\, a(n+99) - 17\, a(n+100) + a(n+101) = 0
\end{aligned}
$$