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A SURVEY OF THE MAXIMUM DEPTH PROBLEM FOR INDECOMPOSABLE EXACT COVERS*

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1. THE PROBLEM

Let U be a finite set and let $\mathscr{P}(U)$ denote the collection of all subsets of U. By a system of blocks on U, we shall mean a function $f\colon \mathscr{P}(U) \to \{0, 1, 2, \ldots\}$. We may thing of $f(S)$ as the number of times the subset S appears as a block in the system. Thinking of systems in terms of functions leads naturally to concept of addition of systems:

$$(f + g)(S) = f(S) + g(S) .$$

We may think of $f + g$ as the system obtained by superimposing the systems f and g.

We say that the system f has *weight* $w[f]$ where

$$w[f] = \sum_{S} f(S) .$$

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We say that f is an *exact cover of depth* $d[f]$ if for each $u \in U$,

$$\sum_{u \in S} f(S) = d[f] .$$

Clearly if f and g are exact covers then $f+g$ is an exact cover. In fact $d[f+g] = d[f] + d[g]$ and $w[f+g] = w[f] + w[g]$.

Let z denote the zero function. We say that an exact cover f is *decomposable* if $f = g_1 + g_2$ where for each i, g_i is an exact cover and $g_i \neq z$. If f is not decomposable, we say f is *indecomposable.* Let $|U| = n$ and define

$$D_n = \max \{d[f]: f \text{ is an indecomposable exact cover of } U\} ,$$

where it is understood that $D_n = \infty$ is a possibility.

The obvious question to ask is if D_n is finite. This question was answered in the affirmative by Huckemann and Jurkat [3]. Although many people have considered the problem, the exact value of D_n is known only if $n < 6$, and the best known upper and lower bounds are quite "divergent" from one another. It is hoped that a wider interest in this problem will produce some better results.

2. EXAMPLES

For $|U| = 1$ there are exactly two non-zero indecomposable exact cover, namely f and g defined by $f(\phi) = 0$, $f(U) = 1$, $g(\phi) = 1$, $g(U) = 0$. Since $d[f] = 1$ while $d[g] = 0$, $D_1 = 1$. Now let $U = \{u_1, u_2\}$. One easily verifies that f, defined by $f(\phi) = f(\{u_1\}) = f(\{u_2\}) = 0$, $f(U) = 1$, g defined by $g(\phi) = 1$, $g(\{u_1\}) = g(\{u_2\}) = g(U) = 0$, and h defined by $h(\phi) = h(U) = 0$, $h(\{u_1\}) = h(\{u_2\}) = 1$, are the only indecomposable exact covers of U. Hence $D_2 = 1$. In general, for any finite set U and any partition of U, the function which takes on the value 1 at each cell of the partition and zero elsewhere is an indecomposable exact cover of depth 1. If $U = \{u_1, u_2, u_3\}$, there is only one indecomposable exact cover aside from those arising from partitions or the empty set:

$$f(S) = \begin{cases} 1 & \text{if } |S| = 2\,, \\ 0 & \text{if } |S| \neq 2\,. \end{cases}$$

This is easily verified, and hence $D_3 = 2$.

In general, for an n-set U, f defined by

(1) $$f(S) = \begin{cases} 1 & \text{if } |S| = n-1\,, \\ 0 & \text{if } |S| \neq n-1\,, \end{cases}$$

is an indecomposable exact cover of U of depth $n-1$. The indecomposability of this exact cover can be easily proved directly; however, it will follow at once from the following Proposition.

For an arbitrary system of blocks f on U we define $\bar{f}$, the *complement of* f, by $\bar{f}(S) = f(\bar{S})$ for all $S \subseteq U$, where $\bar{S}$ denotes the complement of S in U. We observe that

(2) $$\bar{\bar{f}} = f\,,$$

(3) $$\overline{f+g} = \bar{f} + \bar{g}\,,$$

(4) $$\bar{f} = z \text{ iff } f = z\,,$$

Proposition 1. *$\bar{f}$ is an (indecomposable) exact cover if and only if f is an (indecomposable) exact cover. Furthermore,* $d[f] + d[\bar{f}] = w[\bar{f}] = w[f]$.

Proof. It is clear that $w[\bar{f}] = w[f]$. Assume that f is an exact cover and let $u \in U$. Then,

$$\sum_{u \in S} \bar{f}(S) = \sum_{u \in S} f(\bar{S}) = \sum_{u \notin S} f(S) =$$
$$= \sum_{S} f(S) - \sum_{u \in S} f(S) = w[f] - d[f]\,.$$

It follows at once that $\bar{f}$ is an exact cover of depth $w[f] - d[f]$. Now assume that f is indecomposable and that $\bar{f} = g_1 + g_2$, where g_1 and g_2 are exact covers. Using the above implication along with (2) and (3) above, we have $f = \bar{g}_1 + \bar{g}_2$, where $\bar{g}_i$ is an exact cover. Since f is indecomposable $\bar{g}_i = z$ for some i and thus by (4), $g_i = z$ for some i. We conclude that $\bar{f}$ is indecomposable. Finally by applying (2) above the equivalence is established.

Returning to our example, we see that, for any n-set U, f as defined in (1) is the complement of the indecomposable exact cover arising from the partition of U into one element cells and is therefore indecomposable.

If $|U| = 4$, we have, up to permutation, one partition into 4 cells, one partition into 3 cells, two partitions into 2 cells, and one partition with 1 cell. Under complementation, the first type of partition (4 cells) yields f as in (1). The partitions of type 3 (2 cells) are invariant under complementation, and the fourth type (1 cell) yields the exact cover of depth 0 under complementation. The remaining type (3 cells) yields an indecomposable exact cover of depth 2 under complementation. It is not too difficult to show that up to permuation there are only 3 more types of indecomposable exact covers, g and h below and the complement of h (g is self-complementary). Let $U = \{u_1, u_2, u_3, u_4\}$.

$$g(S) = \begin{cases} 1 & \text{if } S = \{u_1\}, \{u_1, u_2, u_3\}, \{u_2, u_4\}, \text{ or } \{u_3, u_4\}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

$$h(S) = \begin{cases} 2 & \text{if } S = \{u_1, u_2, u_3\}, \\ 1 & \text{if } S = \{u_1, u_4\}, \{u_2, u_4\}, \text{ or } \{u_3, u_4\}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Clearly $w[g] = 4$ and $d[g] = 2$ while $w[h] = 5$ and $d[h] = 3$. Hence h and f as defined in (1) have the maximum depth among all indecomposable exact covers of a 4-set, and $D_4 = 3$. A table of indecomposable exact covers of a 4-set occurs in [6].

Beyond 4, the complete listing of indecomposable exact covers becomes tedious – it becomes "impossible" beyond 6 or 7.

The asymptotic behavior of $c_n = \dfrac{D_n}{d_n}$ may, in fact, prove to be a very interesting question.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix}$$

An indecomposable exact cover of depth 4 which is not an extremal exact cover.

Table 3

6. ACKNOWLEDGEMENTS

I would like to thank the many people who have responded to my requests for information with pre- and reprints, unpublished manuscripts, and long informative letters. Most of their names occur in the paper. In particular, I would like to thank Lloyd Shapley for a great deal of help and consideration. Due to the "naturalness" of this problem, it is highly probable that I am not aware of all the work done on exact covers. I would like to offer my apologies to everyone whose work I have overlooked.

REFERENCES

[1] L.J. Billera, Some recent results in n-person game theory, *Math. Prog.*, 1, No. 1 (1971), 58-67.

[2] J. Hadamard, Résolution d'une question relative aux déterminants, *Bull. Sci. Math.*, (2), 17, (1893), 240-246.

[3] F. Huckemann – W.B. Jurkat, Finiteness theorems for coverings and semi-groups, *to appear.*

[4] B. Peleg, An inductive method for constructing minimal balanced collections of finite sets, *Naval Res. Logist. Quart.*, 12, June (1965), 155-162.

[5] J. Shalhevet, *unpublished notes.*

[6] L.S. Shapley, On balanced sets and cores, *Naval Res. Logist. Quart.*, 14 (1967), 453-460.

[7] L.S. Shapley, Balanced sets, Sperner's lemma, and Scarf's theorem on the core, *to appear in Mathematical Programming,* Academic Press, New York, 1973.

[8] L.S. Shapley, *unpublished notes.*

[9] J. Spencer, *unpublished notes.*

[10] J.H. van Lint – H.O. Pollak, An "offense-last-move" game against perfect local defense at targets of arbitrary values, *to appear.*

[11] J. Williamson, Determinants whose elements are 0 and 1, *Am. Math. Monthly,* 53 (1946), 427-434.