A triangle for calculating A181613.

Peter Bala, April 25, 2017

Oste and van der Jeugt [1, Section 7] show that a continued fraction of the form

$$\cfrac{1}{1-xd_0-\cfrac{xh_1}{1-xd_1-\cfrac{xh_2}{1-xd_2-\cfrac{xh_3}{1-xd_3-\cdots}}}} \tag{1}$$

is the generating function for 2-Motzkin paths weighted by the integers d_i and h_i. This combinatorial interpretation allows one to rapidly calculate the terms of a sequence whose generating function can be expressed as a continued fraction of the form (1). The results are conveniently displayed in the form of a lower triangular array, where the $d's$ occur as multiplication factors along diagonals of the array and the $h's$ as horizontal multiplication factors along rows of the array. In the particular case of A181613, which gives the expansion of the Jacobian elliptic function $\mathrm{nc}(x,k)$, the bivariate generating function can be expressed as the continued fraction

$$1/(1-x/(1-2^2(1-k^2)x/(1-3^2x/(1-4^2(1-k^2)x/(1-5^2x/(1-...)))))).$$

So in this case the $d's$ are all zero and the horizontal multiplication factors are given by the formula $h_{2n+1}=(2n+1)^2, h_{2n}=(2n)^2(1-k^2)$. The row polynomials of A181613 appear on the leading diagonal of the following lower triangular array:

$$\begin{array}{lllllll}
1 & & & & & & \\
\downarrow & & & & & & \\
1 & \text{— x1—>} & 1 & & & & \\
\downarrow & & \downarrow & & & & \\
1 & \text{— x}4(1-k^2)\text{—>} & 5-4k^2 & \text{— x1 —>} & 5-4k^2 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \text{— x9 —>} & 14-4k^2 & \text{— x}4(1-k^2)\text{—>} & 61-76k^2+16k^4 & \text{— x1 —>} & 61-76k^2+16k^4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \quad \ddots
\end{array}$$

References

[1] R. Oste and J. Van der Jeugt, Motzkin paths, Motzkin polynomials and recurrence relations, Electronic Journal of Combinatorics 22(2) (2015), #P2.8. Section 7