5. Square-free Words: The Thue–Morse Theorem

The Thue–Morse sequence is an infinite binary sequence that is constructed by starting with 0 and then, at each step, replacing every occurrence of 01 with 0110 and every occurrence of 10 with 1001. The sequence begins:

0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, ...

This sequence is aperiodic, meaning it does not repeat indefinitely. It is also binary, consisting only of 0s and 1s. The Thue–Morse sequence is notable for its self-similarity and its occurrence in various areas of mathematics, including combinatorics on words, ergodic theory, and cell biology.

The Thue–Morse theorem states that the Thue–Morse sequence is aperiodic. This means that there is no block of consecutive terms that repeats itself.