Notes on for A156894

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We show that the supercongruence A156894(p) $\equiv$ A156894(1) $\pmod{p^3}$ holds for all prime $p \ge 5$.

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The terms of A156894 are defined by means of the binomial sum

$$a(n) = \sum_{k=0}^{n} \binom{n}{k} \binom{2n+k-1}{k}. \tag{1}$$

Equivalently,

$$a(n) = [x^n] \left(\frac{1+x}{(1-x)^2} \right)^n. \tag{2}$$

Supercongruences. Given an integer sequence $s(n)$, there exists a unique formal power series $G(x) = 1 + g_1 x + g_2 x^2 + \cdots$, with rational coefficients, such that

$$s(n) = [x^n]\, G(x)^n \quad \text{for } n \ge 1. \tag{3}$$

$G(x)$ is given by

$$G(x) = \frac{x}{\text{Rev } (xE(x))}, \tag{4}$$

where Rev denotes the series reversion (inversion) operator, and the power series $E(x) = \exp\left(\sum_{n \ge 1} s(n) \frac{x^n}{n} \right)$. See [Stan'99, Exercise 5.56 (a), p. 98, and its solution on p. 146] or [Bal'15].

We can invert (4) to express $E(x)$ in terms of $G(x)$:

$$E(x) = \frac{1}{x} \text{Rev } \left(\frac{x}{G(x)} \right). \tag{5}$$

One simple consequence of (4) and (5) is the following:

the power series $G(x)$ is integral $\iff$ the power series $E(x)$ is integral.

For an integer sequence $s(n)$, the condition that the power series $E(x) = \exp\left(\sum_{n\geq 1} s(n)\frac{x^n}{n}\right)$ is integral is known to be equivalent to the statement that the Gauss congruences

$$s\left(mp^k\right) \equiv s\left(mp^{k-1}\right) \pmod{p^k}$$

hold for all prime p and positive integers m, k [Stan'99, Ex. 5.2 (a), p. 72, and its solution on p. 104].

It therefore follows from (2) and the above remarks that the sequence $a(n) = \text{A156894}(n)$ satisfies the Gauss congruences. In fact, calculation suggests that A156894 satisfies the stronger supercongruences

$$a\left(mp^k\right) \equiv a\left(mp^{k-1}\right) \pmod{p^{3k}} \tag{6}$$

for prime $p \geq 5$ and all positive integers m and k. We prove a particular case.

Proposition 1. *The supercongruence $a(p) \equiv 3 \pmod{p^3}$ holds for prime $p \geq 5$.*

Proof. Let $p \geq 5$ be prime. We make use of the binomial sum representation (1) for $a(p)$. We rewrite the sum by separating out the first $(k = 0)$ summand and last $(k = p)$ summand and adding together the k-th and $(p-k)$-th summands for $1 \leq k \leq \frac{p-1}{2}$ to obtain

$$a(p) = 1 + \binom{3p-1}{p} + \sum_{k=1}^{\frac{p-1}{2}} \binom{p}{k}\left(\binom{2p+k-1}{k} + \binom{3p-k-1}{p-k}\right).$$

Now by [Mes'11, equation 35]

$$\binom{3p-1}{p} = \frac{2}{3}\binom{3p}{p} \equiv 2 \pmod{p^3} \quad \text{for prime } p \geq 5.$$

Hence

$$a(p) \equiv 3 + \sum_{k=1}^{\frac{p-1}{2}} \binom{p}{k}\left(\binom{2p+k-1}{k} + \binom{3p-k-1}{p-k}\right) \pmod{p^3}, \quad \text{prime } p \geq 5. \tag{7}$$

To establish the Proposition we will show that each summand on the right side of (7) is divisible by p^3. Clearly, the first factor $\binom{p}{k}$ in each summand is divisible by p for k in the range of summation. Therefore, to prove the Proposition, it is enough to show that the second factor $\binom{2p+k-1}{k} + \binom{3p-k-1}{p-k}$ is divisible by p^2 for all values of k in the range of summation. To show this, we write the second factor as a product of two terms, each of which is divisible by p.

One easily checks that

$$\binom{2p+k-1}{k}+\binom{3p-k-1}{p-k}=\frac{(2p+k-1)!}{(p-k)!(2p-1)!}\left\{\frac{(3p-k-1)!}{(2p+k-1)!}+\frac{(p-k)!}{k!}\right\}. \tag{8}$$

The first factor on the right side of (7) is a rational number divisible by p for k in the range $1...\frac{p-1}{2}$, since its numerator is exactly divisible by p^2 and its denominator is exactly divisible by p. To show that the second factor on the right side of (8) is also divisible by p we first set $r = p-2k \geq 1$ and $s = p-2k-1 \geq 0$. Then we have

$$\begin{aligned}\frac{(3p-k-1)!}{(2p+k-1)!}+\frac{(p-k)!}{k!} &= (3p-k-1)\cdots(3p-k-r)\ +(p-k)\cdots(p-k-s)\\ &\equiv (-1)^r(k+1)\cdots(k+r)+(-1)^{s+1}k\cdots(k+s)\ (\bmod p)\\ &\equiv 0\ (\bmod p)\end{aligned}$$

since $r = s+1$ and $k + r \equiv -k \pmod p$.

We have shown that $\binom{2p+k-1}{k}+\binom{3p-k-1}{p-k}$ is divisible by p^2 for $1 \leq k \leq \frac{p-1}{2}$, thus completing the proof of the Proposition. ☐

A generalisation. We define a 3-parameter family of sequences $a_{(r,i,j)}(n)$ by

$$a_{(r,i,j)}(n) = [x^{rn}]\left(\frac{(1+x)^i}{(1-x)^j}\right)^n \qquad r \in \mathbb{N},\ i,j \in \mathbb{Z}. \tag{9}$$

So by (2), $a_{(1,1,2)}(n) = \text{A156894}(n)$. Other sequences of this type already in the OEIS include $a_{(1,1,1)}(n) =$ A002003(n), $a_{(2,1,1)}(n) =$ A103885(n) and $a_{(1,2,1)}(n) =$ A119259(n). We conjecture that the supercongruences

$$a_{(r,i,j)}\left(np^k\right) \equiv a_{(r,i,j)}\left(np^{k-1}\right)\ \left(\bmod p^{3k}\right) \tag{10}$$

hold for prime $p \geq 5$ and all $r, n, k \in \mathbb{N}$ and $i, j \in \mathbb{Z}$.

References

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[Mes'11] R. Mestrovic, Wolstenholme's theorem: Its Generalizations and Extensions in the last hundred and fifty years (1862-2011), arXiv:1111.3057 [math.NT], 2011.

[Stan'99] R. P. Stanley, Enumerative Combinatorics, Volume 2, Cambridge University Press, 1999