# Proof of conjectured recurrence for A154146

Robert Israel

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A154146 is the set of nonnegative integers $n$ such that $16 + n(n+1)/2$ is a square. If that square is $y^2$ and $x = 2n + 1$, this says

$$x^2 = 8y^2 - 127 \tag{1}$$

We must solve this Pell-type Diophantine equation. Note that all integer solutions of (1) have $x$ odd, so $n = (x-1)/2$ will be a member of A154146 if and only if $(x, y)$ is a solution in positive integers of (1).

The matrix

$$M = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}$$

leaves invariant the quadratic form $Q(x, y) = x^2 - 8y^2$, so if $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution of (1), so is

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 8y \\ x + 3y \end{pmatrix}$$

Since both $M$ and

$$M^{-1} = \begin{pmatrix} 3 & -8 \\ -1 & 3 \end{pmatrix}$$

have integer entries, $M \begin{pmatrix} x \\ y \end{pmatrix}$ is an integer solution if and only if $\begin{pmatrix} x \\ y \end{pmatrix}$ is.

The curve $x^2 = 8y^2 - 127$ is a hyperbola, and multiplication by $M$ maps the branch in the upper half plane to itself. Moreover, it increases the $x$ coordinate of points on this branch. One point on that branch is $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$, which $M$ maps to $\begin{pmatrix} 29 \\ 11 \end{pmatrix}$. Thus all positive integer solutions of (1) are of the form $M^k \begin{pmatrix} x \\ y \end{pmatrix}$ where $k$ is a nonnegative integer and $\begin{pmatrix} x \\ y \end{pmatrix}$ is an integer solution with $y > 0$ and $0 \le x \le 29$. By explicit enumeration we find that there are just two such solutions: $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 29 \\ 11 \end{pmatrix}$. Thus the positive integer solutions are

$\begin{pmatrix} x_n \\ y_n \end{pmatrix}$, $n \geq 0$, where

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 29 \\ 11 \end{pmatrix}, \quad \begin{pmatrix} x_{n+2} \\ y_{n+2} \end{pmatrix} = M \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Since the characteristic polynomial of $M$ is $\lambda^2 - 6\lambda + 1$, we find that $x_{n+4} - 6x_{n+2} + x_n = 0$. With $x_n = 2a_n + 1$, this translates to $a_{n+4} - 6a_{n+2} + a_n = 2$, and thus

$$a_{n+5} - a_{n+4} - 6a_{n+3} + 6a_{n+2} + a_{n+1} - a_n = (a_{n+5} - 6a_{n+3} + a_{n+1}) - (a_{n+4} - 6a_{n+2} + a_n) = 0$$

which is the conjectured recurrence.