The C 4 lattice and a continued fraction for log(2)

Peter Bala, March 14 2024

A142993: the crystal ball sequence for the lattice C_4. A142993(n) := P(n) is the polynomial function

 $P(n) = (2*n + 1)^2 * (4*n^2 + 4*n + 3)/3.$

A CAS such as Maple can be used to evaluate the series

Sum_{k = 0..n} 1/((k + 1)*P(k)*P(k+1)) = 17/12 - 2*log(2). The purpose of this note is to convert the series to a continued fraction.

Recall the fundamental 3-term recurrences for the numerator and denominator of a continued fraction. If we write the finite continued fraction

in the form

$$R(n) = A(n)/B(n)$$

then A(n) and B(n) are polynomials in a(i), b(j), that, for $n \ge 2$, satisfy the 3-term recurrences

A(n) = b(n) * A(n-1) + a(n) * A(n-2)

$$B(n) = b(n) * B(n-1) + a(n) * B(n-2)$$

with initial values

$$A(1)/B(1) = a(1)/b(1)$$

and

$$A(2)/B(2) = a(1)*b(2)/(b(1)*b(2) + a(2)).$$

Returning to A142993, we define sequences $\{A(n) : n \ge 0\}$ and $\{B(n) : n \ge 0\}$ by

 $A(n) = B(n) * Sum_{k = 0..n} 1/((k + 1)*P(k)*P(k+1))$

B(n) = P(n+1) * (2*n + 2)!,

so that $A(n)/B(n) = Sum_{k = 0..n} 1/((k + 1)*P(k)*P(k+1))$. The sequence {B(n)} is clearly integral; it will turn out that {A(n)} is also integral. The first few values are

n		0	1	2	3
• •	•				2744352 90357120

We show that $\{A(n)\}$ and $\{B(n)\}$ satisfy the same 3-term recurrence. Firstly, it is easy to check that B(n) satisfies the 3-term recurrence

 $u(n) = 2*(4*n^2 + 4*n + 33)*u(n-1) - 4*n^2*(4*n^2 - 1)*u(n-2).$ We show that A(n) satisfies the same recurrence (thus showing that A(n) is an integer). By definition

$$A(n) = B(n) * Sum_{\{k = 0..n\}} 1/((k + 1) * P(k) * P(k+1)).$$

Hence

$$A(n+1) = B(n+1) * Sum_{k} = 0..n+1 \frac{1}{(k+1)} * P(k) * P(k+1))$$

= B(n+1) * Sum_{k} = 0..n} 1/((k+1) * P(k) * P(k+1))
+ B(n+1)/((n+2) * P(n+1) * P(n+2))
= (B(n+1)/B(n)) * A(n) + B(n+1)/((n+2) * P(n+1) * P(n+2))

Substituting B(n) = P(n+1)*(2*n+2)! and multiplying both sides of the resulting identity by (n+2)*P(n+1) we find that

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(n+2) * P(n+1) * A(n+1) = (n+2) * (2*n+3) * (2*n+4) * P(n+2) * A(n)
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+
$$(2*n + 4)!$$
 ... (1)

Hence

$$(n+3) * P(n+2) * A(n+2) = (n+3) * (2*n+5) * (2*n+6) * P(n+3) * A(n+1)$$

+ $(2*n + 6)!$... (2)

Multiplying (1) by (2*n + 5)*(2*n + 6) and subtracting from (2) and then replacing n with n - 2 we find after a short calculation that A(n) satisfies the same 3-term recurrence as satisfied by B(n):

 $A(n) = 2*(4*n^2 + 4*n + 33)*A(n-1) - 4*n^2*(4*n^2 - 1)*A(n-2).$ The first few coefficients of the recurrence are shown below.

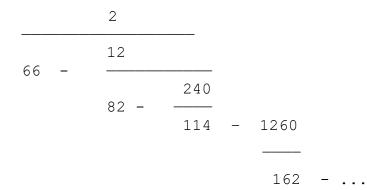
n	1	2	3	4	
4*n^2*(4*n^2 - 1)	12	240	1260	4032	
2*(4*n^2 + 4*n + 33)	82	114	162	226	
It then follows from the	fundamen	tal r	recurrenc	es sat:	isfied by

the numerators and denominators of a continued fraction that the series

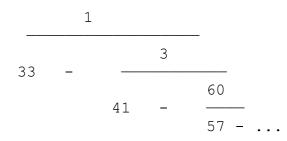
Sum $\{k > = 0\}$ 1/((k + 1)*P(k)*P(k+1))

= Limit $\{n \rightarrow 00\} A(n)/B(n)$

has the continued fraction expansion



By means of an equivalence transformation this is equal to the continued fraction



with partial numerators and partial denominators (after the first)

equal to $n^2 (4*n^2 - 1)$ and $2*(4*n^2 + 4*n + 33)$ (= $2*((2*n + 1)^2 + 2*(4^2))$) respectively.

A similar result holds for the crystal ball sequences of the other C_n lattices.