The expected value of the spectral radius of a 2×2 matrix, whose entries are independent random variables, uniformly distributed over $[0, 1]$

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Let

$$A = \begin{bmatrix} X & Y \\ Z & U \end{bmatrix},$$

where X, Y, Z, U are independent random variables, uniformly distributed over $[0, 1]$. The eigenvalues of the matrix A are

$$\frac{X + U \pm \sqrt{(X - U)^2 + 4YZ}}{2}$$

and, consequently, the spectral radius (as a matter of fact, in this case the largest eigenvalue) is

$$r = \frac{1}{2}[X + U + \sqrt{(X - U)^2 + 4YZ}].$$

From here we obtain that the expected value $E(r)$ of r is given by

$$E(r) = \frac{1}{2}[E(X) + E(U) + E[\sqrt{(X - U)^2 + 4YZ}]$$

or

$$E(r) = \frac{1}{2} + \frac{1}{2}J, \tag{1}$$

where

$$J = \int\int\int\int_{\Omega} \sqrt{(x - u)^2 + 4yz}\, dx\, dy\, dz\, du, \tag{2}$$

Ω being the four-dimensional cube $\{(x, y, z, u) : 0 \leq x, y, z, u \leq 1\}$.

We denote $f(x, y, z, u) = \sqrt{(x - u)^2 + 4yz}$. First we integrate with respect to x between 0 and 1. In a straightforward way we obtain

$$\int_0^1 f dz = \frac{1}{6y}([(x - u)^2 + 4y]^{3/2} - |x - u|^3). \tag{3}$$

Now we integrate with respect to u between 0 and 1. In the integral of the first term we perform the substitution $u - x = t, du = dt$ and we make use of the formula

$$\int (x^2 + a^2)^{3/2} dx = \frac{3}{8} a^4 \ln(x + \sqrt{x^2 + a^2}) + \frac{3}{8} a^2 x (a^2 + x^2)^{1/2} + \frac{1}{4} x (a^2 + x^2)^{3/2}$$

while in the integral of $|x - u|^3$ we break up the interval $[0, 1]$ of the variable u into the intervals $[0, x]$ and $[x, 1]$, where $u \le x$ and $u \ge x$, respectively. We obtain

$$\int_0^1 \int_0^1 f \, dz \, du = g_1(x, y) + g_2(x, y) + g_3(x, y) + g_4(x, y) + g_5(x, y), \qquad (4)$$

where

$$g_1(x, y) = y \ln(1 - x + \sqrt{(1 - x)^2 + 4y}),$$

$$g_2(x, y) = -y \ln(-x + \sqrt{x^2 + 4y}),$$

$$g_3(x, y) = \frac{1}{24y} (1 - x)[(1 - x)^2 + 4y]^{1/2} [(1 - x)^2 + 10y],$$

$$g_4(x, y) = \frac{1}{24y} x (x^2 + 4y)^{1/2} (x^2 + 10y),$$

$$g_5(x, y) = -\frac{1}{24y} [x^4 + (1 - x)^4].$$

Integrating (4) with respect to x, we obtain

$$\int_0^1 \int_0^1 \int_0^1 f \, dx \, dz \, du = h_1(y) + h_2(y) + h_3(y) + h_4(y) + h_5(y), \qquad (5)$$

where $h_j(y) = \int_0^1 g_j(x, y) \, dx \quad (j = 1, ..., 5)$. The substituion $1 - x = t$ yields

$$h_1(y) = y \int_0^1 \ln(t + \sqrt{t^2 + 4y}) \, dt$$

and now

$$h_1(y) + h_2(y) = y \int_0^1 \ln \frac{x + \sqrt{x^2 + 4y}}{-x + \sqrt{x^2 + 4y}} \, dx = 2y \int_0^1 \ln(x + \sqrt{x^2 + 4y}) \, dx - y \ln(4y).$$

Integrating by parts, we obtain

$$h_1(y) + h_2(y) = 2y \ln(1 + \sqrt{1 + 4y}) - 2y\sqrt{1 + 4y} + 4y\sqrt{y} - y \ln(4y). \qquad (6)$$

If in the expression of $h_3(y)$ we use the substitution $1 - x = t$, then we obtain

$$h_3(y) = \frac{1}{24y} \int_0^1 t(t^2 + 10y) \sqrt{t^2 + 4y} \, dt = h_4(y)$$

Performing the substitution $t^2 + 4y = s^2$, we obtain

$$h_3(y) = h_4(y) = \frac{1}{24y} \int_{2\sqrt{y}}^{\sqrt{1+4y}} s^2(s^2 + 6y)\, ds$$

and an elementary computation leads to

$$h_3(y) = h_4(y) = \frac{1}{120y}[(1 + 4y)^{3/2}(1 + 14y) - 112y^{5/2}]. \tag{7}$$

For $h_5(y)$ we obtain in a straightforward manner

$$h_5(y) = \frac{1}{60y}. \tag{8}$$

Introducing (6),(7), and (8) into (5), we obtain

$$\int_0^1 \int_0^1 \int_0^1 f\, dx\, dz\, du = 2y \ln(1 + \sqrt{1 + 4y} + \frac{32}{15} y^{3/2} - y \ln(4y) \\ + \frac{3}{10}\sqrt{1 + 4y} - \frac{16}{15} y\sqrt{1 + 4y} + \frac{1}{60} \frac{\sqrt{1 + 4y} - 1}{y}.$$

The integrals of these terms are elementary, leading to

$$J = \int_0^1 \int_0^1 \int_0^1 \int_0^1 f\, dx\, dy\, dz\, du = \frac{29}{30} \ln \frac{1 + \sqrt{5}}{2} - \frac{43\sqrt{5}}{360} + \frac{1429}{1800}.$$

Introducing this into (1), we obtain

$$E(r) = \frac{29}{60} \ln \frac{1 + \sqrt{5}}{2} - \frac{43\sqrt{5}}{720} + \frac{3229}{3600} \approx 0.9959872.$$