

# CHEBYSHEV SERIES EXPANSION OF THE ELLIPTIC INTEGRAL OF THE SECOND KIND

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ABSTRACT. The Chebyshev series expansion of the (Incomplete) Elliptic Integral of the Second Kind  $E(m, \varphi)$  is evaluated.

## 1. INTRODUCTION AND SCOPE

**Definition 1.** *The Chebyshev polynomials  $T_n(x)$  are even or odd functions of  $x$  defined as [1, (22.3.6)][4, (3.6)]*

$$(1) \quad T_0(x) = 1, \quad T_n(x) = \frac{n}{2} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m}, \quad n = 1, 2, 3 \dots$$

where the Gauss bracket  $\lfloor \cdot \rfloor$  denotes the largest integer not greater than the number it embraces.

The polynomials are orthogonal over the interval  $[-1, 1]$  with weight function  $1/\sqrt{1-x^2}$  [1, (22.2.4)][7, (4.2)][2]

$$(2) \quad \int_{-1}^1 T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m \neq 0, \\ 0, & n \neq m. \end{cases}$$

Elliptic Integrals of the Second Kind [6, 3, 15, 17, 10, 11] are:

**Definition 2.** *(Elliptic Integrals of the second kind) [1, 17.2.9]*

$$(3) \quad E(m, \varphi) \equiv \int_0^\varphi (1 - m \sin^2 \vartheta)^{1/2} d\vartheta.$$

**Definition 3.** *(Complete Elliptic Integrals of the second kind)*

$$(4) \quad E(m) \equiv E(m, \pi/2).$$

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*Date:* February 16, 2013.

*2000 Mathematics Subject Classification.* Primary 33C45, 42C20; Secondary 41A50, 41A10; Tertiary 65D15.

*Key words and phrases.* Chebyshev series, orthogonal polynomials, Elliptical Integral.

2. FOURIER-CHEBYSHEV EXPANSION OF  $(1 - m \sin^2 \varphi)^{1/2}$ 

The integrand of (3) is expanded in a binomial series [1, (3.6.11)] and written with the aid of [13, 1.320.1] and [1, (22.3.15)]

$$(5) \quad (1 - m \sin^2 \varphi)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-m)^n \sin^{2n} \varphi$$

$$= \sum_{n=0}^{\infty} \binom{1/2}{n} (-m)^n \frac{1}{2^{2n}} \binom{2n}{n} + \sum_{n=1}^{\infty} \binom{1/2}{n} (-m)^n \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-)^{n-k} \binom{2n}{k} T_{2n-2k}(\cos \varphi).$$

In the first term, the binomials are converted to Pochhammer Symbols [18, 19] and the power series in  $m$  is recognized as a Gauss Hypergeometric Function [1, (17.3.10), (15.3.15), (15.4.7)]

$$(6) \quad \sum_{n=0}^{\infty} \binom{1/2}{n} (-m)^n \frac{1}{2^{2n}} \binom{2n}{n} = {}_2F_1 \left( -\frac{1}{2}, \frac{1}{2}; 1; m \right) = \frac{2}{\pi} E(m).$$

**Remark 1.** Because the power series converges poorly if  $m \rightarrow 1$ , expansion around  $m = 1$  is often considered instead, which is written as

$$(7) \quad E(m) \equiv E_1(\eta) - E_2(\eta) \ln \eta; \quad \eta \equiv 1 - m,$$

where  $E_1$  and  $E_2$  are power series of their argument [1, (15.3.11), (17.3.36)] [6, 14].

**Remark 2.** The hypergeometric series [1, 17.3.12]

$$(8) \quad E(m) = \frac{\pi}{2} {}_2F_1 \left( -\frac{1}{2}, \frac{1}{2}; 1; m \right),$$

is well known. The coefficients are sequence A010370 of the OEIS [20].

The second term in (5) is re-summed with  $\alpha \equiv n - k$ ,

$$(9) \quad \sum_{n=1}^{\infty} \binom{1/2}{n} (-m)^n \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-)^{n-k} \binom{2n}{k} T_{2n-2k}(\cos \varphi)$$

$$= \sum_{\alpha=1}^{\infty} \sum_{k=0}^{\infty} \binom{1/2}{\alpha+k} (-m)^{\alpha+k} \frac{1}{2^{2(\alpha+k)-1}} (-)^{\alpha} \binom{2(\alpha+k)}{k} T_{2\alpha}(\cos \varphi)$$

$$= \sum_{\alpha=1}^{\infty} m^{\alpha} \frac{1}{2^{2\alpha-1}} T_{2\alpha}(\cos \varphi) \underbrace{\sum_{k=0}^{\infty} \binom{1/2}{\alpha+k} (-m)^k \frac{1}{2^{2k}} \binom{2\alpha+2k}{k}}_{\frac{\Gamma(3/2)}{\Gamma(\alpha+k+1)\Gamma(\frac{3}{2}-\alpha-k)} \left(-\frac{m}{4}\right)^k \frac{\Gamma(2\alpha+2k+1)}{k!\Gamma(2\alpha+k+1)}}$$

$$= \sum_{\alpha=1}^{\infty} m^{\alpha} \frac{1}{2^{2\alpha-1}} T_{2\alpha}(\cos \varphi) \binom{1/2}{\alpha} {}_2F_1 \left( \alpha - 1/2, \alpha + 1/2; 2\alpha + 1; m \right)$$

$$= 2 \sum_{\alpha=1}^{\infty} \binom{1/2}{\alpha} \left( \frac{m}{4} \right)^{\alpha} T_{2\alpha}(\cos \varphi) {}_2F_1 \left( \alpha - 1/2, \alpha + 1/2; 2\alpha + 1; m \right).$$

Plugging this and (6) into (5) gives the Fourier-Chebyshev expansion

$$(10) \quad \sqrt{1 - m \sin^2 \varphi} = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; m\right) T_0(\cos \varphi) \\ + 2 \sum_{\alpha=1}^{\infty} \binom{1/2}{\alpha} \left(\frac{m}{4}\right)^{\alpha} {}_2F_1\left(\alpha - \frac{1}{2}, \alpha + \frac{1}{2}; 2\alpha + 1; m\right) T_{2\alpha}(\cos \varphi) \\ = 2 \sum'_{\alpha=0}^{\infty} \binom{1/2}{\alpha} \left(\frac{m}{4}\right)^{\alpha} {}_2F_1\left(\alpha - \frac{1}{2}, \alpha + \frac{1}{2}; 2\alpha + 1; m\right) T_{2\alpha}(\cos \varphi).$$

The tic mark at the sum symbol means that the contribution of the term build by the summation index at zero is halved before added.

Because this is the first derivative of  $E$ , we conclude from numerical tests that  $dE/d\varphi$  is accurate to about

$$(11) \quad \begin{aligned} &6 \times 10^{-19} \text{ at } \varphi = 0.6, m = 0.2 \\ &1 \times 10^{-12} \text{ at } \varphi = 0.6, m = 0.5 \\ &3 \times 10^{-7} \text{ at } \varphi = 0.6, m = 0.8 \end{aligned}$$

with 12 terms ( $\alpha \leq 12$ ) in this series. (The first numbers in this table are relative accuracies).

### 3. TAYLOR SERIES OF THE INCOMPLETE ELLIPTIC INTEGRAL

The integrated binomial expansion of (5) is

$$(12) \quad E(m, \varphi) = \int_0^{\varphi} \sqrt{1 - m \sin^2 \vartheta} d\vartheta = \sum_{n=0}^{\infty} \binom{1/2}{n} (-m)^n \int_0^{\varphi} \sin^{2n} \vartheta d\vartheta \\ = \sum_{n=0}^{\infty} \binom{1/2}{n} (-m)^n \left[ \frac{1}{2^{2n}} \binom{2n}{n} \varphi + \frac{(-)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-)^k \binom{2n}{k} \frac{\sin(2n-2k)\varphi}{2n-2k} \right].$$

It is a Taylor expansion in  $m$  and a Fourier expansion in  $\varphi$ . The term at  $n = 0$  (the lowest order term corresponding to  $m = 0$ ) is just  $\varphi$ .

**Remark 3.** In numerical practise, partial integration leads to a recursive version of the representation [5]:

$$(13) \quad 2n \int_0^{\varphi} \sin^{2n} \theta d\theta = -\sin^{2n-1} \varphi \cos \varphi + (2n-1) \int_0^{\varphi} \sin^{2n-2} \theta d\theta.$$

The pre-integrated term may also be evaluated recursively: At  $n = 1$ ,  $\sin \varphi \cos \varphi = \frac{1}{2} \sin(2\varphi)$ . And multiplication with  $\sin^2 \varphi$  is a multiplication by  $[1 - \cos(2\varphi)]/2$ .

The power series of  $E(m, \varphi)$  is generated by repeated differentiation of the nested function  $\frac{\partial E}{\partial \varphi} = \sqrt{1 - m \sin^2 \varphi}$ . Defining  $\sqrt{1 - mx^2} = F(x)$ ,  $x = \sin \varphi$ , the derivative of the outer function is [13, 0.430.1][16]

$$(14) \quad \frac{d^n F(\varphi)}{d\varphi^n} = \frac{U_1^{(n)}}{1!} F'(x) + \frac{U_2^{(n)}}{2!} F''(x) + \dots + \frac{U_n^{(n)}}{n!} F^{(n)}(x).$$

In this case

$$(15) \quad F(x) = \sum_{n=0}^{\infty} \binom{1/2}{n} (-mx^2)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\partial^n F}{\partial x^n} \Big|_{x=0}$$

with power series coefficients [20, A079484]

$$(16) \quad F^{(n)}(x) = \begin{cases} 0 & , n \text{ odd} \\ -\frac{m^{n/2}[(n-1)!!]^2}{n-1} & , n=2,4,6,8,\dots \end{cases} = \begin{cases} -m & n=2; \\ -3m^2 & n=4; \\ -45m^3 & n=6; \\ \dots & \dots \end{cases}$$

The inner derivatives are [13, 0.430]

(17)

$$U_k^{(n)} = \frac{d}{d\varphi^n} \sin^k \varphi - \frac{k}{1!} \sin \varphi \frac{d^n}{d\varphi^n} \sin^{k-1} \varphi + \frac{k(k-1)}{2!} \sin^2 \varphi \frac{d^n}{d\varphi^n} \sin^{k-2} \varphi + \dots + (-)^{k-1} k \sin^{k-1} \varphi \frac{d^n}{d\varphi^n} \sin \varphi.$$

Because  $F^{(k)}(\varphi) = 0$  for odd  $k$ , we are only interested in even  $k$ . Because the aim is to calculate the derivatives (14) in the limit  $\varphi \rightarrow 0$ , we effectively are only calculating the first term of (17) because the powers of  $\sin \varphi$  remove contributions from the others:

$$(18) \quad U_k^{(n)} = \frac{d}{d\varphi^n} \sin^k \varphi.$$

For  $k > n$ , this is zero in the limit  $\varphi \rightarrow 0$ , because  $\sin \varphi$  is proportional to  $\varphi$  at small  $\varphi$ . For  $k \leq n$  we insert  $\sin \varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$  in (18) and again execute the limit for even  $k$ ,

$$(19) \quad \frac{d}{d\varphi^n} \sin^k \varphi = \frac{1}{2^k (-)^{k/2}} \frac{d}{d\varphi^n} \sum_{l=0}^k \binom{k}{l} (-)^{k-l} e^{i(2l-k)\varphi} \\ \rightarrow \frac{(-)^{k/2+n/2}}{2^k} \sum_{l=0}^k \binom{k}{l} (-)^l (2l-k)^n = U_k^{(n)}.$$

We're also only interested in even  $n$ , because  $F$  is an even function of  $\varphi$ , and the contributions of odd  $n$  to (14) vanish.

$$(20) \quad \frac{d}{d\varphi^n} (1 - mx^2)^{1/2} = \frac{U_2^{(n)}}{2!} \underbrace{F^{(2)}(x)}_{-m} + \frac{U_4^{(n)}}{4!} \underbrace{F^{(4)}(x)}_{-3m^2} + \frac{U_6^{(n)}}{6!} \underbrace{F^{(6)}(x)}_{-45m^3}$$

**Example 1.**

(21)

$$\frac{d}{d\varphi^4} (1 - mx^2)^{1/2} = \frac{U_2^{(4)}}{2!} \underbrace{F^{(2)}(x)}_{-m} + \frac{U_4^{(4)}}{4!} \underbrace{F^{(4)}(x)}_{-3m^2} = -\frac{8}{2}(-m) + \frac{24}{4!}(-3m^2) = 4m - 3m^2$$

and this is inserted into the Taylor expansion of  $E$ ,

$$(22) \quad E(m, \varphi) = m\varphi - \frac{m}{3!}\varphi^3 + \frac{1}{5!} \underbrace{(4m - 3m^2)}_{\dots} \varphi^5 + \dots$$

where the coefficient in front of  $\partial^n E(m, \varphi) / \partial \varphi^n$  is determined by  $d^{n-1} F(\varphi) / d\varphi^{n-1}$ .

TABLE 1. Table of the first coefficients  $t_{n,s}$  of the Taylor expansion  $E(m, \varphi) = \sum_{n=1,3,5,\dots}^{\infty} \left( \sum_{s=0}^{(n-1)/2} t_{n,s} m^s \right) \frac{\varphi^n}{n!}$ .

$n \setminus s$	0	1	2	3	4	5	6
1	1						
3	0	-1					
5	0	4	-3				
7	0	-16	60	-45			
9	0	64	-1008	2520	-1575		
11	0	-256	16320	-105840	189000	-99225	
13	0	1024	-261888	4055040	-15800400	21829500	-9823275
15	0	-4096	4193280	-149909760	1153152000	-3178375200	3575672100
17	0	16384	-67104768	5459650560	-79048569600	390486096000	-829555927200
19	0	-65536	1073725440	-197553991680	5250196224000	-44028669484800	158114177020800
21	0	262144	-17179803648	7128049582080	-342926948044800	4733629310592000	-27127279769990400
23	0	-1048576	274877644800	-256867482009600	22196806410240000	-494967328786022400	4374957588627225600
25	0	4194304	-4398045462528	9251352517017600	-1429585972125696000	50895131682447360000	-678995658312136704000
27	0	-16777216	70368739983360	-333114661294571520	91817299554140160000	-5179577084488654848000	102813992833509974016000
$n \setminus s$	7	8	9	10			
15	-1404728325						
17	786647862000	-273922023375					
19	-272809478541600	223520371074000	-69850115960625				
21	76080985108128000	-110419063310556000	79629132195112500	-22561587455281875			
23	-18791074088301715200	43103072303587296000	-53956699975408230000	34744844681134087500			
25	4308669456480829440000	-14698645956937182240000	28473254229447027000000	-31325951964510493290000			
27	-941595592608878118912000	4603043908989680394240000	-12973489089359818219200000	21727241901915587037000000			
$n \setminus s$	11	12	13				
23	-9002073394657468125						
25	18220196550786715485000	-4348001449619557104375					
27	-21317629964420457117450000	11304803769010848471375000	-2500100833531245335015625				

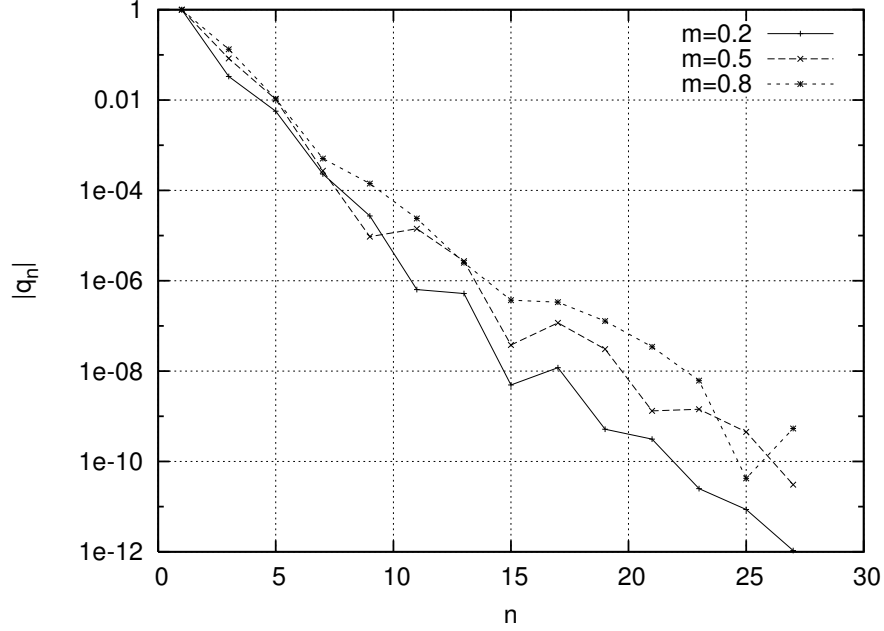


FIGURE 1. The absolute value  $|q_n|$  of the coefficients in the Taylor expansion  $E(m, \varphi) = \sum_{n=1,3,5,\dots}^{\infty} q_n(m) \varphi^n$ .

The results of the expansion of  $E$  in this bivariate power series of  $m$  and  $\varphi$  can be phrased as the Table 1 of integers, equivalent to sequence A120362 of the OEIS [20], or discussed in terms of the diminishing of the expansion coefficients for increasing powers as in Figure 1.

#### 4. TAYLOR SERIES OF THE INVERSE

**4.1. Inverse Complete Elliptic Integral.** Equation (8) proposes a power series inversion of the form

$$(23) \quad m = \sum_{j=0}^{\infty} M_j [L(m)]^j,$$

where  $L(m) \equiv \frac{2}{\pi} E(m) - 1$ .  $E$  is in the range  $1 \leq E(m) \leq \pi/2$ , so  $L$  is in the range  $0 \leq L(m) \leq 0.5708$ . The low-order coefficients  $M_L$  are gathered in Table 2.

We plug this ansatz into

$$(24) \quad \sqrt{1 - m \sin^2 \varphi} = \sum_{l=0}^{\infty} \binom{1/2}{l} (-m)^l \sin^{2l} \varphi,$$

$$= \sum_{l=0}^{\infty} \binom{1/2}{l} (-)^l (M_0 + M_1 L(m) + M_2 L^2(m) + M_3 L^3(m) + \dots)^l \sin^{2l} \varphi,$$

TABLE 2. Coefficients  $M_j$  of  $m = \sum_{j=0}^{\infty} M_j [L(m)]^j$  according to (23).

$j$	
1	$-4 \approx -4.000000\text{e}+00$
2	$-3 \approx -3.000000\text{e}+00$
3	$1/2 \approx 5.000000\text{e}-01$
4	$-5/8 \approx -6.250000\text{e}-01$
5	$15/16 \approx 9.375000\text{e}-01$
6	$-203/128 \approx -1.585938\text{e}+00$
7	$373/128 \approx 2.914062\text{e}+00$
8	$-11637/2048 \approx -5.682129\text{e}+00$
9	$47445/4096 \approx 1.158325\text{e}+01$
10	$-200211/8192 \approx -2.443982\text{e}+01$
11	$1736661/32768 \approx 5.299869\text{e}+01$
12	$-61615957/524288 \approx -1.175231\text{e}+02$
13	$278370715/1048576 \approx 2.654750\text{e}+02$
14	$-1277419515/2097152 \approx -6.091211\text{e}+02$
15	$1485167693/1048576 \approx 1.416366\text{e}+03$
16	$-447159319511/134217728 \approx -3.331597\text{e}+03$
17	$2124891688311/268435456 \approx 7.915838\text{e}+03$
18	$-20374453075051/1073741824 \approx -1.897519\text{e}+04$
19	$49224941414375/1073741824 \approx 4.584430\text{e}+04$
20	$-1916238411650325/17179869184 \approx -1.115398\text{e}+05$

and write this as a power series of  $L(m)$  by expanding all the  $(\dots)^l$

$$(25) \quad = 1 - \binom{1/2}{1} \sin^2 \varphi L(m) + \left[ -\binom{1/2}{1} \sin^2 \varphi + \binom{1/2}{2} \sin^4 \varphi \right] L^2(m) + \dots L^3(m)$$

where the coefficients are polynomials in  $\sin^2 \varphi$ . This is then integrated over  $\varphi$ , individually for each power of  $L(m)$ , and represents an expansion

$$(26) \quad E(m, \varphi) = \underbrace{\Phi_0(\varphi)}_{\varphi} + \Phi_1(\varphi)L(m) + \Phi_2(\varphi)L^2(m) + \Phi_3(\varphi)L^3(m) + \dots$$

**4.2. Numerical.** A purely numeric approach to calculating the inverse (which is calculating  $\varphi$  for a given  $E$ ) would be a Newton Method [12] based on

$$(27) \quad \frac{\partial}{\partial \varphi} E = \sqrt{1 - m \sin^2 \varphi},$$

$$(28) \quad \frac{\partial^2}{\partial \varphi^2} E = -\frac{m \sin \varphi \cos \varphi}{\sqrt{1 - m \sin^2 \varphi}} = -\frac{m \sin(2\varphi)}{2\partial E / \partial \varphi},$$

searching for a zero of

$$(29) \quad E(m, \varphi) - E = 0 \quad (= f(\varphi)),$$

for a given  $E$  by iterations

$$(30) \quad \varphi^{(n+1)} = \varphi^{(n)} - \frac{f}{f'} - \frac{1}{2} \frac{f^2 f''}{f'^3} = \varphi^{(n)} - \frac{E(m, \varphi) - E}{\sqrt{1 - m \sin^2 \varphi}} + \frac{m [E(m, \varphi) - E]^2 \sin \varphi \cos \varphi}{2(1 - m \sin^2 \varphi)^2}$$

The second order Newton term is obtained by ignoring the  $f'''$  in the Taylor approximation around  $E(m, \varphi) \approx E$ .

**4.3. Analytical.** The power series of  $\varphi(E)$  can be build by iterative differentiation of

$$(31) \quad \frac{d\varphi}{dE} = \frac{1}{dE/d\varphi} = \frac{1}{\sqrt{1 - m \sin^2 \varphi}}$$

with the Bürmann-Lagrange series. We illustrate the construction of the first three terms.

$$(32) \quad \frac{d\varphi}{dE} = \frac{1}{\sqrt{1 - m \sin^2 \varphi}} \rightarrow 1 \quad (\varphi = 0).$$

$$(33) \quad \frac{d^2\varphi}{dE^2} = -\frac{1}{2} \frac{1}{(1 - m \sin^2 \varphi)^{3/2}} \cdot (-2m \sin \varphi) \cos \varphi \cdot \underbrace{\frac{d\varphi}{dE}}_1 = \underbrace{\frac{m \sin \varphi \cos \varphi}{(1 - m \sin^2 \varphi)^{3/2}}}_{\equiv \square} \frac{d\varphi}{dE} \rightarrow 0 \quad (\varphi \rightarrow 0).$$

$$(34) \quad \begin{aligned} \frac{d^3\varphi}{dE^3} &= \underbrace{\frac{d}{dE} \square}_{\frac{d}{d\varphi} \square \frac{d\varphi}{dE}} \frac{d\varphi}{dE} + \square \frac{d^2\varphi}{dE^2} = \frac{\partial}{\partial \varphi} \left[ \frac{m \sin \varphi \cos \varphi}{(1 - m \sin^2 \varphi)^{3/2}} \right] \frac{d\varphi}{dE} \cdot \frac{d\varphi}{dE} + \square \frac{d^2\varphi}{dE^2} \\ &= m \underbrace{\left[ \frac{\cos^2 \varphi - \sin^2 \varphi}{()^{3/2}} - \frac{3 \sin \varphi \cos \varphi}{2 ()^{5/2}} \right]}_{\rightarrow 1} \underbrace{\left( \frac{d\varphi}{dE} \right)^2}_{\rightarrow 1} + \underbrace{\square}_{\rightarrow 0} \cdot \underbrace{\frac{d^2\varphi}{dE^2}}_{\rightarrow 0} \rightarrow m \quad (\varphi \rightarrow 0). \end{aligned}$$

The procedure continues in the manner outlined for nested derivatives in Section 3. The coefficients are illustrated in Table 3.

The poor convergence of this process for  $m$  close to unity is illustrated in Figure 2. The inverse function obviously has a singularity at  $m = 1$ .

## 5. CHEBYSHEV SERIES OF THE INTEGRAL KERNEL

The Chebyshev expansion of  $E(m, \varphi)$  as a function of  $\varphi$  is developed by introducing the Chebyshev expansion of the kernel, and integrating this in a post-processing step.

$$(35) \quad \sqrt{1 - m \sin^2 \varphi} = \sum_{n=0}^{\infty} a_n T_n(2\varphi/\pi) \rightsquigarrow \sum_{n=0,2,4,\dots}^{\infty} a_n = \sqrt{1 - m} \quad \text{if } \varphi = \pi/2.$$

Projection of  $a_n$  with the orthogonality of (2) yields

$$(36) \quad a_n = \left( \frac{2}{\pi} \right)^2 \int_{-\pi/2}^{\pi/2} \sqrt{1 - m \sin^2 \varphi} \frac{T_n(\frac{2\varphi}{\pi})}{\sqrt{1 - (\frac{2\varphi}{\pi})^2}} d\varphi = \frac{2}{\pi} \int_0^\pi \sqrt{1 - m \sin^2 \frac{\pi \cos \vartheta}{2}} \cos(n\vartheta) d\vartheta.$$



TABLE 3. Table of the first coefficients  $\hat{t}_{n,s}$  of the Taylor expansion  $\varphi = \sum_{n=1,3,5,\dots}^{\infty} \left( \sum_{s=0}^{(n-1)/2} \hat{t}_{n,s} m^s \right) \frac{[E(m, \varphi)]^n}{n!}$ .

$n \setminus s$	0	1	2	3	4	5	6
1	1						
3	0	1					
5	0	-4	13				
7	0	16	-284	493			
9	0	-64	4944	-31224	37369		
11	0	256	-81088	1406832	-5165224	4732249	
13	0	-1024	1306880	-56084992	474297712	-1212651548	901188997
15	0	4096	-20954112	2119725312	-36844921856	197165292576	-384956066148
17	0	-16384	335466496	-78133315584	2632424633600	-26244540079232	100943948724960
19	0	65536	-5368365056	2845340643328	-179667596318720	3136570734804736	-21137162003634560
21	0	-262144	85897838592	-103006683463680	11945842974142464	-351760499405374464	3898447497590416128
23	0	1048576	-1374382981120	3718290677760000	-782156055927521280	37920514321139957760	-664679452662903020544
25	0	-4194304	21990204243968	-134033011163791360	50746026934805790720	-3985638496417094369280	107659947828279779475456
$n \setminus s$	7	8	9	10			
15	240798388357						
17	-159093845666992	85948640603761					
19	62935807354446112	-83075501407668752	39504564917358001				
21	-19496734832646859008	47150268292587304224	-53510172948139694580	22726779729476308093			
23	5237800159265149198080	-20629800009911330814720	41898392885546432895600	-41687821234115043757420			
25	-1284582204423263717179392	7727753098970152111921920	-24966257989286139557396160	43637700434503811923080880			
$n \setminus s$	11	12					
23	15998009117983994065693						
25	-38645100078961318415022632	13526765851190230940840809					

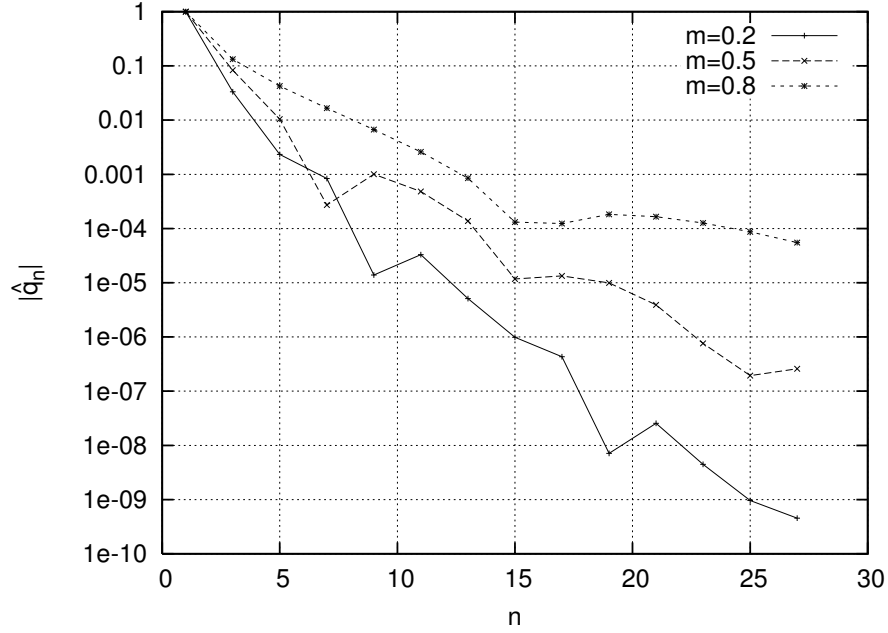


FIGURE 2. The absolute value  $|\hat{q}_n|$  of the coefficients in the Taylor expansion  $\varphi = \sum_{n=1,3,5,\dots}^{\infty} \hat{q}_n(m)[E(m, \varphi)]^n$ .

where we substituted  $2\varphi/\pi = x$  and  $x = \cos \vartheta$  and  $T_n(\cos \vartheta) = \cos(n\vartheta)$  [1, 22.3.15]. As before we proceed with the binomial expansion plus Fourier expansion of odd powers of the sines [13, 1.320.1]

$$\begin{aligned}
 (37) \quad & \sqrt{1 - m \sin^2 \frac{\pi \cos \vartheta}{2}} \\
 &= \sum_{l=0}^{\infty} \binom{1/2}{l} (-m)^l \frac{1}{2^{2l}} \binom{2l}{l} + \sum_{l=1}^{\infty} \binom{1/2}{l} (-m)^l \frac{1}{2^{2l-1}} \sum_{k=0}^{l-1} (-1)^{l-k} \binom{2l}{k} \cos[\pi(l-k) \cos \vartheta]
 \end{aligned}$$

This inserted into (36) produces by the first term

$$(38) \quad \underbrace{\sum_{l=0}^{\infty} \binom{1/2}{l} (-m)^l \frac{1}{2^{2l}} \binom{2l}{l}}_{\text{see above}} \underbrace{\int_0^{\pi} \cos(n\vartheta) d\vartheta}_{\pi \delta_{n,0}} = \pi {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; m\right) \delta_{n,0}.$$

The integration of the second term uses [1, (9.1.21)]

$$(39) \quad \int_0^{\pi} \cos[\pi(l-k) \cos \vartheta] \cos(n\vartheta) d\vartheta = \pi (-1)^{n/2} J_n((l-k)\pi),$$

in full writing...

$$(40) \quad \rightsquigarrow \sum_{l=1}^{\infty} \binom{1/2}{l} (-m)^l \frac{1}{2^{2l-1}} \sum_{k=0}^{l-1} (-1)^{l-k} \binom{2l}{k} \pi (-1)^{n/2} J_n((l-k)\pi).$$

As above this is re-summed with  $l - k \equiv \alpha$

$$(41) \quad = \pi(-)^{n/2} \sum_{\alpha=1}^{\infty} \frac{m^{\alpha}}{2^{2\alpha-1}} J_n(\pi\alpha) \underbrace{\sum_{k=0}^{\infty} \binom{1/2}{\alpha+k} (-m)^k \frac{1}{2^{2k}} \binom{2\alpha+2k}{k}}_{(-)^{\alpha+1} \frac{(1/2)_{\alpha}}{\alpha!(2\alpha-1)!} {}_2F_1(\alpha-\frac{1}{2}, \alpha+\frac{1}{2}; 2\alpha+1; m)}$$

$$= \pi(-)^{n/2+1} \sum_{\alpha=1}^{\infty} \left(-\frac{m}{4}\right)^{\alpha} J_n(\pi\alpha) \frac{(1/2)_{\alpha}}{\alpha!(\alpha-\frac{1}{2})!} {}_2F_1(\alpha-\frac{1}{2}, \alpha+\frac{1}{2}; 2\alpha+1; m).$$

$$(42) \quad \rightsquigarrow a_n = 2\delta_{n,0} \underbrace{{}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; m)}_{2E(m)/\pi} + 2(-)^{1+n/2} \sum_{\alpha=1}^{\infty} \left(-\frac{m}{4}\right)^{\alpha} J_n(\pi\alpha) \frac{(1/2)_{\alpha}}{\alpha!(\alpha-\frac{1}{2})!} {}_2F_1(\alpha-\frac{1}{2}, \alpha+\frac{1}{2}; 2\alpha+1; m).$$

This ought reproduce [1, (17.3.35)], perhaps in conjunction with [1, (15.3.11)]. So far this provides the expansion

$$(43) \quad \sqrt{1 - m \sin^2 \varphi} = \sum_{n=0}^{\infty} a_n T_n(2\varphi/\pi).$$

**Lemma 1.** [7, (4.8)][8, p. 54][9, (2.12)]

$$(44) \quad \int T_n(x) dx = \begin{cases} T_1(x), & n = 0, \\ \frac{1}{4} T_2(x), & n = 1, \\ \frac{1}{2} \left( \frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1} \right), & n > 1, \end{cases}$$

We integrate (43) with this lemma,

$$(45) \quad E(m, \varphi) = \int_0^{\varphi} \sqrt{1 - m \sin^2 \theta} d\theta = \frac{a_0}{2} \varphi + \sum_{n=2,4,6,\dots} a_n \int_0^{\varphi} T_n\left(\frac{2\varphi}{\pi}\right) d\varphi$$

$$= \frac{a_0}{2} \varphi + \sum_{n=2,4,6,\dots} a_n \frac{\pi}{2} \int_0^{2\varphi/\pi} T_n(x) dx$$

$$= \frac{a_0}{2} \underbrace{\varphi}_{\frac{\pi}{2} T_1(2\varphi/\pi)} + \sum_{n=2,4,6,\dots} a_n \frac{\pi}{4} \left[ \frac{T_{n+1}(2\varphi/\pi)}{n+1} - \frac{T_{n-1}(2\varphi/\pi)}{n-1} \right]$$

$$= \frac{\pi}{4} \left\{ a_0 T_1\left(\frac{2\varphi}{\pi}\right) + \sum_{n=2,4,\dots} a_n \left[ \frac{T_{n+1}(2\varphi/\pi)}{n+1} - \frac{T_{n-1}(2\varphi/\pi)}{n-1} \right] \right\}.$$

This formally completes the Chebyshev representation, although a more condensed representation of  $a_n$  than the infinite series (42) is desirable.

## REFERENCES

1. Milton Abramowitz and Irene A. Stegun (eds.), *Handbook of mathematical functions*, 9th ed., Dover Publications, New York, 1972. MR 0167642 (29 #4914)
2. R. Barrio, *Algorithms for the integration and derivation of Chebyshev series*, Appl. Math. Comp. **150** (2004), 707–717. MR MR2039669 (2005a:65019)
3. Roland Bulirsch, *Numerical calculation of elliptic integrals and elliptic functions*, Num. Math. **7** (1965), no. 1, 78–90. MR 0175284

4. Yu. G. Bulychev and E. Yu. Bulycheva, *Some new properties of the Chebychev polynomials and their use in analysis and design of dynamic systems*, Autom. Remote Control **64** (2003), no. 4, 554–563. MR 2092602
5. Paul F. Byrd and Morris D. Friedman, *Handbook of elliptical integrals for engineers and physicists*, 2nd ed., Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. LXVII, Springer, Berlin, Göttingen, 1971. MR MR0277773 (43 #3506)
6. W. J. Cody, *Chebyshev approximations for the complete elliptic integrals  $K$  and  $E$* , Math. Comp. **19** (1965), no. 89, 105–112. MR 0171370
7. ———, *A survey of practical rational and polynomial approximation of functions*, SIAM Rev. **12** (1970), no. 3, 400–423. MR 42 #2627
8. Leslie Fox and Ian Bax Parker, *Chebyshev polynomials in numerical analysis*, Oxford mathematical handbooks, Oxford University Press, Oxford, 1968.
9. W. Fraser, *A survey of methods of computing minimax and near-minimax polynomial approximations for functions of a single independent variable*, J. ACM **12** (1965), no. 3, 295–314. MR 0182125
10. Toshio Fukushima, *Fast computation of jacobian elliptic functions and incomplete elliptic integrals for constant values of elliptic parameter and elliptic characteristic*, Celest. Mech. Dyn. Astr. **105** (2009), 245–260. MR 2551836
11. ———, *Precise and fast computation of the general complete elliptic integral of the second kind*, Math. Comp. **80** (2011), 1725–1743. MR 2785476
12. ———, *Numerical inversion of a general incomplete elliptic integral*, J. Comput. Appl. Math. **237** (2013), no. 1, 43–61.
13. I. Gradstein and I. Ryshik, *Summen-, Produkt- und Integraltafeln*, 1st ed., Harri Deutsch, Thun, 1981. MR 0671418 (83i:00012)
14. Cecil Hastings (ed.), *Approximations for digital computers*, Princeton University Press, Princeton, 1955.
15. D. J. Hofsommer and R. P. van de Riet, *On the numerical calculation of elliptic integrals of the first and second kind and the elliptic functions of jacobi*, Numer. Math. **5** (1963), no. 1, 291–302. MR 0166906
16. Warren P. Johnson, *The curious history of Faà di Bruno's formula*, Amer. Math. Monthly **109** (2002), no. 3, 217–234. MR 1903577 (2003d:01019)
17. Tohru Morita and Tsuyoshi Horiguchi, *Convergence of the arithmetic-geometric mean procedure for the complex variables and the calculation of the complete elliptic integrals with complex modululs*, Num. Math. **20** (1972), no. 5, 425–430. MR 0315870
18. Ranjan Roy, *Binomial identities and hypergeometric series*, Amer. Math. Monthly **94** (1987), no. 1, 36–46. MR 0873603 (88f:05012)
19. Lucy Joan Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966. MR 0201688
20. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, <http://oeis.org/>. MR 1992789 (2004f:11151)

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