Cyclotomic polynomials and the generating functions of A118205 and A118206

Peter Bala, April 09 2023

The generating functions of A118205 and A118206 are the infinite products

$$\mathrm{G118205}(x) = \prod_{n\ge 1} (1 - x^n)^{\lambda(n)}, \tag{1}$$

$$\mathrm{G118206}(x) = \prod_{n\ge 1} (1 - x^n)^{-\lambda(n)}, \tag{2}$$

where

$$\lambda(n) = \sum_{d^2|n} \mu\left(\frac{n}{d^2}\right)$$

is the Liouville function [3].

We find alternative expressions for these generating functions as infinite products of cyclotomic polynomials. All our products are formal products.

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The n-th cyclotomic polynomial $C_n(x)$ is equal to the rational fraction

$$C_n(x) = \prod_{d|n} \left(1 - x^d\right)^{\mu\left(\frac{n}{d}\right)} \quad \text{for } n \ge 2, \tag{3}$$

with $C_1(x) = x - 1$ (see, for example, [2]), where $\mu(n)$ is the Möbius function and the product is taken over the divisors d of n.

Proposition 1. *The generating function of A118205 is* $-\prod_{n\ge 1} C_n\left(x^n\right)$

Proof. We have by (3)

$$
\begin{aligned}
-\prod_{n\geq 1} C_n\left(x^n\right) &= (1-x)\prod_{n\geq 2}\left(\prod_{d|n}\left(1-x^{nd}\right)^{\mu\left(\frac{n}{d}\right)}\right)\\
&= (1-x)\prod_{N\geq 2}\left(\prod_{\substack{N=nd\\ d\,|\,n}}\left(1-x^{N}\right)^{\mu\left(\frac{N}{d^2}\right)}\right)\\
&= (1-x)\prod_{N\geq 2}\left(\prod_{d^2|N}\left(1-x^{N}\right)^{\mu\left(\frac{N}{d^2}\right)}\right)\\
&= \prod_{N\geq 1}\left(1-x^{N}\right)^{\sum_{d^2|N}\mu\left(\frac{N}{d^2}\right)}\\
&= \prod_{N\geq 1}\left(1-x^{N}\right)^{\lambda(N)}\\
&= \mathrm{G118205}(x)\quad \text{by (1).}
\end{aligned}
$$

□

We recall the following properties of the cyclotomic polynomials ([1], p. 160).

Let p be a prime number and n be a positive integer. Then

$$
C_{pn}(x)=\begin{cases} C_n\left(x^p\right) & \text{if } p\mid n \qquad (A)\\[2ex] \dfrac{C_n\left(x^p\right)}{C_n(x)} & \text{if } p\nmid n \qquad (B)\end{cases}
$$

The particular case of (B) when $p=2$ is

$$
C_{2n}(x)=\frac{C_n\left(x^2\right)}{C_n(x)}\quad \text{for } n \text{ odd.} \tag{4}
$$

By a simple induction argument using (A), we find that

$$
C_{2^k n}(x)=C_{2n}\left(x^{2^{k-1}}\right)\quad \text{for } n \text{ odd}, k\geq 1. \tag{5}
$$

Hence, by (4),

$$C_{2^k n}(x) = \frac{C_n\left(x^{2^k}\right)}{C_n\left(x^{2^{k-1}}\right)} \quad \text{for } n \text{ odd, } k \geq 1. \tag{6}$$

In (6), replace x with $x^{2^{k-1}}$ to give

$$C_{2^k n}\left(x^{2^{k-1}}\right) = \frac{C_n\left(x^{2^{2k-1}}\right)}{C_n\left(x^{2^{2k-2}}\right)} \quad \text{for } n \text{ odd, } k \geq 1. \tag{7}$$

In (7), replace x with x^2 to give

$$C_{2^k n}(x^{2^k}) = \frac{C_n\left(x^{2^{2k}}\right)}{C_n\left(x^{2^{2k-1}}\right)} \quad \text{for } n \text{ odd, } k \geq 1. \tag{8}$$

Proposition 2. *The generating function of A118206 is* $\prod_{n\geq 1} C_{2n}\left(x^n\right)$.

Proof. We have

$$\begin{aligned}
\prod_{n\geq 1} C_{2n}\left(x^n\right) &= \prod_{n \text{ odd}} C_{2n}\left(x^n\right) C_{4n}\left(x^{2n}\right) C_{8n}\left(x^{4n}\right) C_{16n}\left(x^{8n}\right)\cdots \\
&= \prod_{n \text{ odd}} \left(\frac{C_n\left(x^{2n}\right)}{C_n\left(x^n\right)}\right)\left(\frac{C_n\left(x^{8n}\right)}{C_n\left(x^{4n}\right)}\right)\left(\frac{C_n\left(x^{32n}\right)}{C_n\left(x^{16n}\right)}\right)\left(\frac{C_n\left(x^{128n}\right)}{C_n\left(x^{64n}\right)}\right)\cdots \quad \text{using (7)} \\
&= -\prod_{n \text{ odd}} \frac{1}{C_n\left(x^n\right)}\left(\frac{C_n\left(x^{2n}\right)}{C_n\left(x^{4n}\right)}\right)\left(\frac{C_n\left(x^{8n}\right)}{C_n\left(x^{16n}\right)}\right)\left(\frac{C_n\left(x^{32n}\right)}{C_n\left(x^{64n}\right)}\right)\cdots \\
&= -\prod_{n \text{ odd}} \frac{1}{C_n\left(x^n\right)}\frac{1}{C_{2n}\left(x^{2n}\right)}\frac{1}{C_{4n}\left(x^{4n}\right)}\frac{1}{C_{8n}\left(x^{8n}\right)}\cdots \quad \text{using (8)} \\
&= -\prod_{n\geq 1}\frac{1}{C_n\left(x^n\right)} \\
&= \text{G118206}(x),
\end{aligned}$$

since the g.f. of A118206 is the reciprocal of the g.f. of A118205, which is equal to $-1/\prod_{n\geq 1} C_n\left(x^n\right)$ by Proposition 1. □

References

[1] T. Nagell, "The Cyclotomic Polynomials" and "The Prime Divisors of the Cyclotomic Polynomial." §46 and 48 in Introduction to Number Theory. New York: Wiley, pp. 158-160 and 164-168, 1951.

[2] Wikipedia, Liouville function

[3] Wikipedia, Cyclotomic polynomial