A note on A094061

Peter Bala, Feb 13 2022

$a(n)$ = the number of n-moves paths of a king starting and ending at the origin of an infinite chessboard.

We say that a sequence $(s(n))_{n\ge1}$ has the *Gauss property* if the congruences

$$s\left(np^k\right) \equiv s\left(np^{k-1}\right) \left(\text{mod } p^k\right)$$

hold for all prime p and positive integers n and k.

We show that A094061 has the Gauss property. We require the following result

$$a(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T(k)^2 \tag{1}$$

contributed by Seiichi Manyama. Here $T(n)$ = A002426(n) denotes the n-th central trinomial number. In words, (1) says that $a(n)$ is the inverse binomial transform of $T(n)^2$.

We also need the following result [1]: given an integer sequence $(s(n))_{n\ge1}$, there exists a unique formal power series $G(x) = 1 + g_1x + g_2x^2 + \cdots$, with rational coeffcients g_i, such that $s(n) = [x^n]\, G(x)^n$ for $n \ge 1$. Then we have

(A) the coefficients of $G(x)$ are integers iff the sequence $(s(n))_{n\ge1}$ has the Gauss property.

The central trinomial number $T(n)$ is defined as $T(n) = [x^n]\left(1 + x + x^2\right)^n$. Hence, by (A), the sequence $(T(n))$ has the Gauss property. Clearly, the sequence $(T(n)^2)$ then also has the Gauss property. It follows from (A) that there is a power series with integer coefficients, $H(x)$ say, such that $T(n)^2 = [x^n]\, H(x)^n$. It is straightforward to see using (1) that

$$a(n) = [x^n]\,(H(x) - x)^n$$

and, since the power series $H(x) - x$ has integer coefficients, a final application of (A) shows that the sequence $(a(n))$ has the Gauss property.

Calculation suggests that in fact A094061 satisfies the stronger supercongruences $a(np^k) \equiv a(np^{k-1}) \left(\text{mod } p^{2k}\right)$ for all prime $p \ge 5$ and positive integers n and k.

[1] Peter Bala, Notes on A156894