Odd harmonic series: Formulas

Harmonic series: (1) $H(m) = \sum_{j=1}^{m} \frac{1}{j}$ (def.)

Odd harmonic series: (2a) $OH(m) = \sum_{j=0}^{m} \frac{1}{2j+1}$ (def.)

(2b) $OH(m) = H(2m+1) - ½H(m)$ (evident)

(3) $H(m) = H_4(m) + O\left(\frac{1}{m^6}\right)$ with $H_4(m) = \log(m) + \gamma + \frac{1}{2m} - \frac{1}{12m^2} + \frac{1}{120m^4}$ (well-known)

(4) $OH(m) = A(m) + B(m) + O\left(\frac{1}{m^6}\right)$ with (see 2b)

$A(m) = \log(2m+1) + \gamma - ½\log(m) - ½\gamma - log\left(1 + \frac{1}{2m}\right) = ½(\log(4m) + \gamma)$

$B(m) = log\left(1 + \frac{1}{2m}\right) + \frac{1}{2\cdot(2m+1)} - \frac{1}{4m} - \frac{1}{12(2m+1)^2} + \frac{1}{24m^2} + \frac{1}{120(2m+1)^4} - \frac{1}{240m^4}$

(5) $F(x) = B(1/(2x)) = log(1+x) + \frac{x}{2\cdot(1+x)} - \frac{x}{2} - \frac{x^2}{12(1+x)^2} + \frac{x^2}{8} + \frac{x^4}{120(1+x)^4} - \frac{x^4}{15}$

Taylor: $F(x) = x - \frac{11x^2}{12} + x^3 - \frac{127x^4}{120} + O(x^5)$

$B(m) = F(1/(2m)) = \frac{1}{2m} - \frac{11}{48m^2} + \frac{1}{8m^3} - \frac{127}{1920m^4} + O\left(\frac{1}{m^5}\right)$

<u>Formula 1</u>: $OH(m) = OH_3(m) - \frac{127\cdot t}{1920m^4}, 0 < t < 1, with$

$OH_3(m) = ½(\log(4m) + \gamma) + \frac{1}{2m} - \frac{11}{48m^2} + \frac{1}{8m^3}$

<u>Formula 2</u>: $OH(m) = ½\left(\log\left(4m + 4 + \frac{1}{6m}\right) + \gamma\right) + O\left(\frac{1}{m^3}\right)$

Proof:

$½\log\left(4m + 4 + \frac{1}{6m}\right) = ½\log(4m) + ½\log\left(1 + \frac{1}{m} + \frac{1}{24m^2}\right)$

$= \log(2) + ½\log(m) + ½\left(\frac{1}{m} - \frac{11}{24m^2}\right) + O\left(\frac{1}{m^3}\right)$ (Taylor)

$= ½\log(4m) + \frac{1}{2m} - \frac{11}{48m^2} + O\left(\frac{1}{m^3}\right)$, q.e.d., see formula 1.

With formula 2, we can approximate the inverse of OH:

Let $k = OH(m)$: $\log\left(4m + 4 + \frac{1}{6m}\right) + \gamma = 2k \Rightarrow 4m + 4 + \frac{1}{6m} = e^{2k-\gamma}$

Solution: $m = \frac{y}{4} - 1 - \frac{2}{3y} + O\left(\frac{1}{y^2}\right)$ with $y = e^{2k-\gamma}$. The precision $m = \frac{y}{4} - 1$ will be sufficient. The least integer exceeding m is $\left\lceil \frac{y}{4} - 1 \right\rceil = \left\lfloor \frac{y}{4} \right\rfloor$, as m certainly is no integer.

<u>Formula 3</u> $A092315(n) = \lfloor ¼e^{2n-\gamma} \rfloor$

Formula 3 can be tested with the more precise formula 1.

Formula 1 can be tested numerically by combining it with formula (2b) and evaluating the parameter t: $t(m) = \frac{1920}{127} \cdot m^4 \cdot \left(OH_3(m) - H_4(2m+1) + H_4(m)\right)$

Result: $t(10) = 0.955, t(100) = 0.995, t(1000) = 0.9995, \rightarrow 1.$