On the sequence A085642

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Abstract

We prove the conjecture by Vladeta Jovovic on the sequence A085642 in Sloane's On-Line Encyclopedia of Integer Sequences.

1 Introduction

In the sequence A085642 in Sloane's On-Line Encyclopedia of Integer Sequences [7]

0,1,1,2,3,6,8,12,17,26,35,49,66,92,121, ...

the nth number $a(n)$ counts the number of columns in the character table of the symmetric group S_n that have zero sums; it was submitted 2003 by Yuval Dekel. As stated in the comments to the sequence, shortly afterwards Vladeta Jovovic made the

Conjecture The number $a(n)$ equals the number of partitions of n with at least one part congruent to 2 mod 4.

In this note, we prove this conjecture.

2 A little bit of character theory

We refer to [4], [6] for details about partitions and about characters of the symmetric group S_n.
Consider a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of the integer n. Thus $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_l = n$; for short, we write $\lambda \vdash n$. We call the λ_i's the parts of λ. Moreover for $i \geq 1$, $m_i = m_i(\lambda)$ denotes the number of parts equal to i in λ.
The set of all partitions of n will be denoted by $P(n)$.
We recall that the conjugacy classes of S_n are labelled by partitions, which correspond to the cycle type of the elements in the conjugacy class.
We denote the cycle type of $\sigma \in S_n$ by $c(\sigma)$.

We will also need some character-theoretic results; we refer to [3, Chapter 4] or [2] for details.
For a finite group G we denote by $\mathrm{Irr}(G)$ its set of irreducible complex characters. For $\chi \in \mathrm{Irr}(G)$ we define

$$\nu_2(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2) \ .$$

Theorem 2.1 *[3, p. 58] (Frobenius-Schur) Let G be a finite group, $\chi \in \mathrm{Irr}(G)$. Then $\nu_2(\chi) = 0$, 1 or -1 if χ is non-real, real and the representation realizable over $\mathbb{R}$, or real but the representation not realizable over $\mathbb{R}$, respectively.*

For S_n, the following holds

Theorem 2.2 *[4, sec. 3.4] Any $\chi \in \mathrm{Irr}(S_n)$ comes from an integral representation. In particular, $\nu_2(\chi) = 1$ for all $\chi \in \mathrm{Irr}(S_n)$.*

For a finite group G, let $\psi_2 : G \to \mathbb{Z}$ be defined by

$$\psi_2(g) = |\{h \in G \mid h^2 = g\}| \ ,$$

i.e., $\psi_2(g)$ counts the number of square roots of g in G.
Clearly, ψ_2 is a class function on G.

Theorem 2.3 *[3] Let G be a finite group. Then*

$$\psi_2 = \sum_{\chi \in \mathrm{Irr}(G)} \nu_2(\chi)\chi \ .$$

Hence we deduce

Corollary 2.4 *Let $n \in \mathbb{N}$. Then for S_n we have*

$$\psi_2 = \sum_{\chi \in \mathrm{Irr}(S_n)} \chi \ .$$

Corollary 2.5 *Let $n \in \mathbb{N}$, $\sigma \in S_n$. Then the column of the character table of S_n labelled by $c(\sigma)$ sums to*

$$\sum_{\chi \in \mathrm{Irr}(S_n)} \chi(\sigma) = |\{\tau \in S_n \mid \tau^2 = \sigma\}| \ .$$

Hence the columns counted by $a(n)$ of our sequence correspond to the conjugacy classes of non-squares in S_n.

Remark. See also [8], solution to problem 7.69 (b), or [5, Ex. 11, p. 120] for this connection between the column sums and the counting of square roots.

3 A little bit of combinatorics

Let's look at an element τ of cycle type $\mu = (1^{m_1}, 2^{m_2}, \ldots)$. This squares to the element $\sigma = \tau^2$ of cycle type $\lambda = (1^{m_1+2m_2}, 2^{2m_4}, 3^{m_3+2m_6} \ldots)$. Hence we have

Lemma 3.1 *The partition λ is the cycle type of a non-square if and only if some even part of λ has odd multiplicity.*

Thus we have the following combinatorial description of $a(n)$:

Corollary 3.2 *Let $n \in \mathbb{N}$. Then*

$$a(n) = |\{\lambda \vdash n \mid \exists i : m_{2i}(\lambda) \text{ is odd}\}| \ .$$

Next we show that the set on the right hand side is in bijection with the partitions appearing in the Conjecture.

We want to define an involution on the set of partitions which is motivated by the Glaisher map. First, we write our partitions in a different way.

Instead of writing the parts of λ with their multiplicity (as in the exponential notation for partitions) we collect this information in suitable triples according to the 2-adic decomposition of multiplicities and the 2-power in the parts. Assume b occurs in λ with multiplicity m written in 2-adic decomposition as $m = \sum_{k\in I} 2^{\varepsilon_k}$, with different exponents ε_k. Write $b = 2^j a$ with a odd. Then we turn the m parts b into the set of triples $(2^j, a, 2^{\varepsilon_k})$, with $k \in I$. Doing this for all parts gives us a set Λ of triples which is just another notation for the partition λ.
We now define a map Γ on $P(n)$ by using this notation. Given $\lambda \in P(n)$, let Γ act on Λ by switching all 2-powers in the triple, i.e., $\Gamma(\Lambda)$ is the set of all triples $(2^k, a, 2^j)$ with $(2^j, a, 2^k) \in \Lambda$. Clearly, this is an involution on $P(n)$.

Proposition 3.3 *The restriction of Γ provides a bijection between the set of partitions of n having an odd part and the set of partitions of n where some part has odd multiplicity, respectively.*

Proof. For λ of the first type, there is a triple $(1, a, 2^k) \in \Lambda$. Hence $(2^k, a, 1) \in \Gamma(\Lambda)$, which immediately implies that $2^k a$ occurs with odd multiplicity in $\Gamma(\lambda)$. The converse is also clear. $\diamond$

Remark 3.4 The restriction of Γ to the set of partitions with odd parts only gives the Glaisher bijection to the set of partitions into distinct parts.

Given $m \in \mathbb{N}_0$, we now define a slight variation Θ_m of Γ. For $\lambda \in P(n)$, all parts not divisible by 2^m are fixed by Θ_m, but for a part $b = 2^{j+m} a$ (with a odd, as before) we now write the corresponding triples as $(2^j, 2^m a, 2^k)$ (instead of $(2^{j+m}, a, 2^k)$). Then we let Θ_m switch the 2-powers of triples with middle component $2^m a$. Again, this is clearly an involution on $P(n)$.
For $r \in \mathbb{N}$, let $2^{\nu_2(r)}$ be the largest 2-power dividing r. Then we have:

Proposition 3.5 *The restriction of Θ_m provides a bijection between the set of partitions of n having a part s with $\nu_2(s) = m$ and the set of partitions of n where some part t with $\nu_2(t) \geq m$ has odd multiplicity, respectively.*

Proof. Partitions of the first type are those with a triple of the form $(1, 2^m a, 2^k)$, partitions of the second type are those with a triple $(2^j, 2^m a, 1)$. $\diamond$

Of course, the restriction of Θ_1 (as above) is just the bijection needed in our context, hence we have

Corollary 3.6 *Jovovic' Conjecture is true.*

An alternative proof is easily obtained by computing generating functions of the number of partitions in the complementary sets.
Let

$$\begin{aligned} b_1(n) &= |\{\lambda \vdash n \mid \forall i \in \mathbb{N}_0 : m_{4i+2}(\lambda) = 0\}| \\ b_2(n) &= |\{\lambda \vdash n \mid \forall i \in \mathbb{N} : m_{2i}(\lambda) \text{ is even}\}| \, . \end{aligned}$$

Both these functions have as generating function the product

$$R(q) = \prod_{i=0}^{\infty} \frac{1}{1-q^{2i+1}} \prod_{i=1}^{\infty} \frac{1}{1-q^{4i}} \, .$$

Indeed in the well-known expression for the generating function $P(q)$ for the partition function $p(n)$

$$P(q) = \prod_{i=1}^{\infty} \frac{1}{1-q^i}$$

the factor $\frac{1}{1-q^i}$ accounts for the multiplicity of i in the partitions (see e.g. [1, p.4]). Therefore $R(q)$ is clearly the generating function for $b_1(n)$. Also

$$\prod_{i=1}^{\infty} \frac{1}{1-(q^i)^2}$$

counts partitions where all parts have even multiplicity and thus

$$\prod_{i=1}^{\infty} \frac{1}{1-(q^{2i})^2}$$

counts partitions where all parts are *even* and have even multiplicity. Rewriting

$$\prod_{i=1}^{\infty} \frac{1}{1-q^{4i}} = \prod_{i=1}^{\infty} \frac{1}{1-(q^{2i})^2}$$

we see that $R(q)$ is also the generating function for $b_2(n)$. Thus $b_1(n) = b_2(n)$ for all $n \geq 0$.

References

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