Monoids of Natural Numbers

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Let \( \mathbb{N} \) denote the set of nonnegative integers. If \( A = \{a_1, a_2, \ldots, a_m\} \) is a set of positive integers satisfying \( \gcd(a_1, a_2, \ldots, a_m) = 1 \), then

\[
\langle a_1, a_2, \ldots, a_m \rangle = \left\{ \sum_{j=1}^{m} x_j a_j : x_j \in \mathbb{N} \text{ for each } 1 \leq j \leq m \right\}
\]

is the subset of \( \mathbb{N} \) generated by \( A \). For example,

\[
\langle a, a+1, a+2, a+3, \ldots, 2a-1 \rangle = \{0\} \cup \{a, a+1, a+2, a+3, \ldots\}
\]

and

\[
\langle 2, b \rangle = \{0, 2, 4, \ldots, b-3\} \cup \{b-1, b, b+1, b+2, b+3, \ldots\}
\]

when \( b \geq 3 \) is odd.

A numerical monoid \( S \) is a subset of \( \mathbb{N} \) that is closed under addition, contains 0, and has finite complement in \( \mathbb{N} \). (Most authors use the phrase “numerical semigroup”, but semigroups by definition need not contain 0, hence the usage is puzzling.) The Frobenius number \( f \) of \( S \) is the maximum element in the set \( \mathbb{N} - S \), and the genus \( g \) of \( S \) is the cardinality of \( \mathbb{N} - S \). Therefore

\[
f (\langle a, a+1, a+2, a+3, \ldots, 2a-1 \rangle) = a-1, \quad f (\langle 2, b \rangle) = b-2,
\]

\[
g (\langle a, a+1, a+2, a+3, \ldots, 2a-1 \rangle) = a-1, \quad g (\langle 2, b \rangle) = (b-1)/2
\]

and, more generally [1],

\[
f (\langle a, b \rangle) = (a-1)(b-1) - 1, \quad g (\langle a, b \rangle) = (a-1)(b-1)/2
\]

when \( \gcd(a, b) = 1 \). It is known that \( f + 1 \leq 2g \) always [2, 3]. Table 1 gives all monoids \( S \) with \( 1 \leq f \leq 4 \) or \( 1 \leq g \leq 4 \).

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Table 1. Numerical Monoids with Small Frobenius Number or Genus

<table>
<thead>
<tr>
<th>$f = 1$</th>
<th>$f = 2$</th>
<th>$f = 3$</th>
<th>$f = 4$</th>
<th>$g = 1$</th>
<th>$g = 2$</th>
<th>$g = 3$</th>
<th>$g = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, 3)$</td>
<td>$(3, 4, 5)$</td>
<td>$(4, 5, 6, 7)$</td>
<td>$(5, 6, 7, 8, 9)$</td>
<td>$(2, 3)$</td>
<td>$(3, 4, 5)$</td>
<td>$(4, 5, 6, 7)$</td>
<td>$(5, 6, 7, 8, 9)$</td>
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<tr>
<td></td>
<td>$(2, 5)$</td>
<td>$(3, 5, 7)$</td>
<td>$(2, 5)$</td>
<td>$(3, 5, 7)$</td>
<td>$(4, 6, 7, 9)$</td>
<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(3, 4)$</td>
<td>$(3, 7, 8)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(2, 7)$</td>
<td>$(4, 5, 7)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(4, 5, 6)$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$(3, 5)$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$(2, 9)$</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Define sequences \([4, 5, 6, 7]\)

\[
\{F_n\}^\infty_{n=1} = \{1, 1, 2, 2, 5, 4, 11, 10, \ldots\},
\]

\[
\{G_n\}^\infty_{n=1} = \{1, 2, 4, 7, 12, 23, 39, 67, \ldots\}
\]

by

\[
F_n = \text{(the number of monoids } S \text{ with } f(S) = n),
\]

\[
G_n = \text{(the number of monoids } S \text{ with } g(S) = n)
\]

then Backelin \[8\] showed that

\[
0 < \liminf_{n \to \infty} 2^{-n/2}F_n < \limsup_{n \to \infty} 2^{-n/2}F_n < \infty,
\]

\[
\frac{1}{\sqrt{2}} (2.47) < \lim_{n \equiv 0 \text{ mod } 2} 2^{-n/2}F_n < \frac{1}{\sqrt{2}} (3.3), \quad \frac{1}{\sqrt{2}} (2.5) < \lim_{n \equiv 1 \text{ mod } 2} 2^{-n/2}F_n < \frac{1}{\sqrt{2}} (3.32)
\]

and Bras-Amorós \[5, 9, 10\] conjectured that

\[
\lim_{n \to \infty} \frac{G_{n+1}}{G_n} = \varphi
\]

where \(\varphi = (1 + \sqrt{5})/2 = 1.6180339887\ldots\) is the Golden mean. Tighter bounds are needed for \(F_n\) asymptotics; it has not even been proved that \(G_n\) is increasing.

A monoid is irreducible if it cannot be written as the intersection of two monoids properly containing it \[11\]. A monoid \(S\) is irreducible if and only if \(S\) is maximal (with respect to set inclusion) in the collection of all monoids with Frobenius number \(f(S)\). Irreducible monoids with odd \(f\) are the same as symmetric monoids (for which \(f = 2g - 1\) always); irreducible monoids with even \(f\) are the same as pseudosymmetric monoids (for which \(f = 2(g - 1)\) always). As an example, \(\langle 3, 4 \rangle\) and \(\langle 2, 7 \rangle\) are the two symmetric monoids with Frobenius number 5; \(\langle 4, 5, 7 \rangle\) is the unique
pseudo-symmetric monoid with Frobenius number 6. Another characterization of symmetry and pseudo-symmetry will be given shortly. Define \([4, 12]\)
\[
\{H_n\}_{n=1}^\infty = \{1, 1, 1, 2, 1, 3, 2, 3, 6, 2, 8, \ldots\}
\]
by
\[
H_n = \text{(the number of irreducible monoids } S \text{ with } f(S) = n)\]
then Backelin \([8]\) showed that
\[
0 < \liminf_{n \to \infty} 2^{-n/6}H_n < \limsup_{n \to \infty} 2^{-n/6}H_n < \infty,
\]
\[
\frac{1}{2}(9.36) < \lim_{n \equiv 0 \mod 6} 2^{-n/6}H_n = \frac{1}{\sqrt{2}} \lim_{n \equiv 3 \mod 6} 2^{-n/6}H_n < c.
\]
No finite value \(c\) (as an upper bound for \(H_n\) asymptotics) has been rigorously proved.

0.1. Sets without Closure. A numerical set \(S\) is a subset of \(\mathbb{N}\) that contains 0 and has finite complement in \(\mathbb{N}\). The Frobenius number of \(S\) is, as before, the maximum element in the set \(\mathbb{N} - S\). Nothing has been assumed about additivity so far. Every numerical set \(S\) has an associated atom monoid \(A(S)\) defined by
\[
A(S) = \{n \in \mathbb{Z} : n + S \subseteq S\}.
\]
Clearly \(A(S) \subseteq S\); also \(A(S) = S\) if and only if \(S\) is itself a numerical monoid. The Frobenius number of \(A(S)\) is the same as the Frobenius number of \(S\); thus there is no possible ambiguity when speaking about \(f(S)\). Let
\[
\mathbb{N}_n = \langle n + 1, n + 2, n + 3, \ldots, 2n + 1 \rangle = \{0\} \cup \{n + 1, n + 2, n + 3, \ldots\}
\]
which we already know has Frobenius number \(n\). Given \(n\), which sets \(S\) have \(A(S) = \mathbb{N}_n\)? Table 2 answers the question for \(1 \leq n \leq 5\). For brevity, we give only \(T\), where \(S = T \cup \mathbb{N}_n\) is a disjoint union.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(T \cup \mathbb{N}_n) with Atom Monoid (\mathbb{N}_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>{1}</td>
</tr>
<tr>
<td>(2)</td>
<td>{1}</td>
</tr>
<tr>
<td>(3)</td>
<td>{1, 2}</td>
</tr>
<tr>
<td>(4)</td>
<td>{1, 2}</td>
</tr>
<tr>
<td>(5)</td>
<td>{1, 2, 3}</td>
</tr>
</tbody>
</table>
Define \([13]\)

\[\{P_n\}_{n=1}^{\infty} = \{1, 2, 3, 6, 10, 20, 37, 74, \ldots\}\]

by

\[P_n = \text{(the number of sets } S \text{ with } A(S) = \mathbb{N}_n)\]

then Marzuola & Miller \([14]\) showed that

\[\lim_{n \to \infty} \frac{P_n}{2^n - 1} \approx 0.484451 \pm 0.005.\]

Also, a numerical set \(S\) with Frobenius number \(n\) satisfying

\[x \in S \text{ if and only if } n - x \notin S\]

is symmetric if \(n\) is odd and pseudo-symmetric if \(n\) is even and \(n/2 \notin S\) (we agree to exclude \(x = n/2\) from consideration). The symmetric cases in Table 2 are marked by \(*\). Define \([13]\)

\[\{Q_k\}_{k=1}^{\infty} = \{1, 1, 2, 3, 6, 10, 20, 37, 73, \ldots\}\]

by

\[Q_k = \text{(the number of symmetric sets } S \text{ with } A(S) = \mathbb{N}_{2k-1})\]

then \([14]\)

\[\lim_{k \to \infty} \frac{Q_k}{2^{k-1}} \approx 0.230653 \pm 0.006.\]

It is interesting the \(Q_{k+2}\) is the number of additive 2-bases for \(\{0, 1, 2, \ldots, k\}\), meaning sets \(\Sigma\) that satisfy

\[\Sigma \subseteq \{0, 1, 2, \ldots, k\} \subseteq \Sigma + \Sigma.\]

The asymptotics for the corresponding “anti-atom” problem for pseudo-symmetric sets are identical to the preceding.

**Addendum.** Work continued on the growth of \(G(n)\) \([15, 16]\), culminating with a theorem by Zhai \([17]\):

\[\lim_{n \to \infty} \frac{G(n)}{\varphi^n} \text{ exists and is finite (and is at least } 3.78).\]

No similar progress can be reported for \(F(n)\).

A **more sums than differences** (MSTD) set is a finite subset \(S\) of \(\mathbb{N}\) satisfying \(|S + S| > |S - S|\). The probability that a uniform random subset of \(\{0, 1, \ldots, n\}\) is an MSTD set is provably > 0.000428 and conjecturally \(\approx 0.00045\), as \(n \to \infty\).
Underlying solution techniques [18] resemble those in [16]; the problem itself reminds us of [19].

Given \( \gcd(a, b, c) = 1 \), let \( \tilde{f}(a, b, c) = f((a, b, c)) + a + b + c \). Ustinov [20, 21] proved that, on average, \( \tilde{f}(a, b, c) \) is asymptotic to \( (8/\pi)\sqrt{abc} \). The following probability density function

\[
p(t) = \begin{cases} 
\frac{12}{\pi}\left(\frac{t}{\sqrt{3}} - \sqrt{4 - t^2}\right) & \text{for } \sqrt{3} \leq t \leq 2 \\
\frac{12}{\pi^2}\left[\sqrt{3}t \arccos\left(\frac{t + 3\sqrt{t^2 - 4}}{4\sqrt{t^2 - 3}}\right) + \frac{3}{2}\sqrt{t^2 - 4}\ln\left(\frac{t^2 - 4}{t^2 - 3}\right)\right] & \text{for } t > 2
\end{cases}
\]

describes more fully the behavior of \( \tilde{f}(a, b, c)/\sqrt{abc} \) as \( \max\{a, b, c\} \to \infty \); in particular, the distribution has a sharp peak at mode 2 and has mean

\[
\int_{\sqrt{3}}^{\infty} tp(t)dt = \frac{8}{\pi}.
\]

In words, \( \tilde{f}(a, b, c) \) is the largest positive integer not representable as \( xa + yb + zc \) for positive coefficients \( x, y, z \). This is more convenient for the analysis – based on continued fractions (Porter’s constant [22] appears in [20]) – leading to proof of such limiting results. Let \( \tilde{g}(a, b, c) \) denote the cardinality of all positive integers not representable as \( xa + yb + zc \), \( x > 0, y > 0, z > 0 \). One of Ustinov’s students calculated the average normalized genus to be \( 8/\pi - 64/(5\pi^2) \); we await the proof. Also, what can be said about rates of growth of \( F_{n,k} \) and \( G_{n,k} \), the counts of monoids when the number of generators is fixed to be \( k \)?

References


