A NOTE ON STEPHAN'S CONJECTURE 77

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Recently Stephan [2] posted 117 conjectures based on extensive analysis of the On-line Encyclopedia of Integer Sequences [1]. Here we give a proof of conjecture 77.

Conjecture 77 is concerned with what is called the **same game**. The moves of this game are performed on binary strings. Specifically, let S be a binary n-string with a run of $k > 1$ consecutive identical digits. Then we define a **reduction rule** by removing those consecutive identical digits thus producing an $(n-k)$-string. Strings that can be reduced to null by a sequence of reduction rules in the same game are called **winning strings**. **Losing strings** are those strings that are not winning strings.

Conjecture 77 states that the number of winning strings of length n is $2^n - 2nF_{n-2} - (-1)^n - 1$ where F_n is the n'th Fibonacci number. There are a total of 2^n binary strings of length n. Therefore, the conjecture says that the number of losing strings of length n is $2nF_{n-2}$ (n odd) or $2nF_{n-2}+2$ (n even). For n even, we call the two losing strings which repeat '10' $n/2$ times or '01' n/2 times the trivial losing strings.

The overall structure of our proof is to find F_{n-2} non-trivial losing strings, each of which represents $2n$ different non-trivial losing strings. We will do this by defining a group action on binary strings in order to show that the orbit of a losing string under this action is $2n$. We then find F_{n-2} distinct orbits.

Consider the group $G = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$ (written multiplicatively). Let r be a generator of $\mathbf{Z}/n\mathbf{Z}$ and c a generator of $\mathbf{Z}/2\mathbf{Z}$. Let Q_n be the set of binary n-strings. Then we define an action of G on Q_n by mapping r to rotation of strings

$$a_1a_2\ldots a_n \mapsto a_2a_3\ldots a_na_1$$

and mapping c to complementation of strings. Here complementation produces the string where each element a_i is mapped to 0 if $a_i = 1$ and 1 if $a_i = 0$.

Proposition 1. *Every G-orbit of Q_n consists entirely of winning strings or entirely of losing strings.*

Proof. Assume S is a winning string of length n that begins with 0. Then there is some sequence of removals that reduces S to null, one of which removes the first 0 in S along with at least one other 0. Notice that removing this 0 cannot bring together a run of consecutive identical digits. Therefore, we can choose to save this run of 0's for the end, making its removal the final removal. Similarly, we can choose to save any 0's at the end of the string for our final removal.

Now consider rS. This string resembles S, except that the 0 at the beginning is now on the end. Notice that we can perform moves on rS analogous to those performed on S above except for the last move that takes S to null. This is because we "protect" any 0's on the ends of S. If we follow the same sequence of removals as before, we will be left with a single run of 0's so that rS is also a winning string. This argument can be repeated for S beginning with 1.

It is obvious that cS must also be a winning string. Thus we have that, given a winning string S, cS and rS are winning, proving that any orbit containing a winning string contains only winning strings (c and r generate G). This implies that any orbit containing a losing string contains only losing strings. □

Thus we have losing orbits and winning orbits. Moreover, Proposition 1 gives a useful corollary. Consider a variant of the same game we call the **wraparound game**. In this variant all reduction rules in the same game are valid and we also allow removal of runs of consecutive identical digits that wrap around the end of the string. Thus

$$0010 \mapsto 1$$

is a reduction rule in the wraparound game.

Corollary 2. *S is a winning string in the same game if and only if S is a winning string in the wraparound game.*

Proof. We prove the non-trivial direction by induction on n the length of a winning string. So assume that any winning string in the wraparound game of length $j < n$ is a winning string in the standard same game. Assume S is a winning n-string in the wraparound game. Consider the case where we remove a run of length k that wraps around. Then we can rotate S to r^iS for some integer i so that this run does not wrap around and eliminate it using the standard same game rules. This produces a winning $(n-k)$-string in the wraparound game which by our induction hypothesis is a winning string in the standard same game. Thus r^iS is a winning string in the standard same game so that Proposition 2 implies that S is a winning string in the standard same game. □

For what follows, it is advantageous to use the wraparound game and Corollary 2 allows us to do this. That is, for an n-string we always consider the first digit to be consecutive to the n'th digit so that runs of consecutive identical digits wrap around.

Lemma 3. *Given any non-trivial losing binary n-string S, there is a rotation r^iS that can be reduced to a single digit in the standard same game.*

Proof. Assume S is a non-trivial losing n-string so that S must contain a run of two or more consecutive identical digits (recall that runs may wrap around!). When we remove this run in the wraparound game, the two digits on either side are identical, so we bring them together and remove them and any other identical digits within the run. This brings together two more identical digits which we remove in a similar fashion. Eventually this must terminate with a single digit a_i in S for $1 \leq i \leq n$ by our assumption that S is a losing string.

Consider the rotation $r^{i-1}S$ making a_i the first digit. Because we do not remove a_i, placing it as the first digit ensures that no run we removed in S will wrap around in $r^{i-1}S$. Thus, all the removals performed on S become allowable removals in the standard same game when performed on $r^{i-1}S$. □

We now consider the structure of losing G-orbits. The following two propositions establish that every losing G-orbit of Q_n has size $2n$.

Proposition 4. *Let S be a non-trivial binary n-string. Assume that $S = r^k S$ for some k, $0 < k < n$. Then S is a winning string.*

Proof. Choose the smallest positive k for which $r^k S = S$. Then any two digits separated by k places (mod n) are identical. Because $r^n(S) = S$ it follows by properties of the greatest common denominator that $r^{\gcd(k,n)}(S) = S$. Thus, because of our choice of k, we have that $\gcd(k, n) = k$ and S repeats its first k digits n/k times.

Let the string T of length k be the first k digits of S. If T is a winning string, then it follows that S is a winning string. So assume T is a losing string. Because S is non-trivial, T must also be non-trivial. By Lemma 3 there is a rotation of T, $r^j T$ which can be reduced to a single digit using the standard same game rules. Then the first k digits of $r^j S$ are $r^j T$ which repeat n/k times. Now reduce each copy of $r^j T$ to a single digit, leaving a single run of n/k consecutive identical digits. Finally, we remove this run of digits, proving that $r^j S$, and therefore S, is a winning string. □

Proposition 5. *Let S be a non-trivial binary n-string. Assume that $cr^k S = S$ for some k, $0 < k < n$. Then S is a winning string.*

Proof. Let k be the smallest positive number for which $cr^k S = S$. Then the first k digits of S are the complement of the next k digits. Similarly, the next k digits must be the complement of these, meaning they are the same as the first k digits. Therefore, the first $2k$ digits of S will be repeated over the length of S. If $n > 2k$, then this pattern is repeated at least twice and by the previous result S must be a winning string. So we are only concerned with the case where $n = 2k$, and the pattern appears only once.

We prove this case by induction on n the length of our string S. So assume any non-trivial string S of length $j < n$ composed of a string followed by its complement is a winning string. Let S be a string of length n. Because S is non-trivial, we can rotate it to $r^i S$ for some i so that the first k digits contains a run of consecutive identical digits. Let T be the first k digits of $r^i S$, so that $r^i S$ consists of T followed by cT. Then we can remove a run from T to produce T' and remove that run's complement from cT to produce cT'. This creates a string S' of length less than n composed of T' followed by cT'. If S' is trivial, then $r^i S$ consists of a single run of 1's and a single run of 0's and so it is a winning string. Our induction hypothesis covers the other case. □

Thus we know that the cardinality of the orbit of a losing string S under G is $2n$. Otherwise, S would be a winning string. What remains to be shown is that there are F_{n-2} losing G-orbits of Q_n, excluding the orbit of trivial strings. To this end, we define a transform on binary n-strings.

Definition. Given a binary string S of length n, we define the **indexing string of S**, denoted $I(S)$ as follows. $I(S)$ is a binary string of length n such that for $i < n$, the i'th element of $I(S)$ is 1 if the i'th element of S is different from the $i + 1$'st element of S and 0 if the i'th element of S is the same as the $i + 1$'st element. For $i = n$, we wraparound. That is, the n'th element of $I(S)$ is 1 if the n'th element of S is different from the first element and the condition for 0 is similar.

Notice the following properties of indexing strings. First, for any binary string S, $I(S) = I(cS)$. Second, $I(rS) = rI(S)$. We also have the following proposition.

Proposition 6. *Given S, The number of* 1*'s in the indexing string* $I(S)$ *is even.*

Proof. The number of 1's counts the number of times that S alternates from 0 to 1 or 1 to 0. If S alternates from 0 to 1 it must at some point alternate back from 1 to 0 because we consider S as wrapping around at the end. $\square$

We now define a third game, the **indexing game**. The moves of this game are performed on finite binary strings containing an even number of 1's. Let S be an n-string in the indexing game with a run of $k \geq 1$ consecutive 0's (where the first digit is consecutive to the n'th). Then this run of 0's is flanked by two 1's, one on each side. We define a **reduction rule** by removing the run of 0's and replacing the two flanking 1's with a single zero, producing an $(n-k-1)$-string. A **winning string** in the indexing game is a string in the indexing game that can be reduced to just a run of $k > 1$ 0's.

Proposition 7. *Let T be a binary n-string with an even number of* 1*'s. Then $T = I(S)$ for some binary string S. T is a winning string in the indexing game if and only if S is a winning string in the wraparound game.*

Proof. Let T be defined as above. It is trivial to note that $T = I(S)$ for some binary n-string S. We will reduce $I(S)$ and S simultaneously using corresponding reduction rules. Notice that corresponding to a run of length k in S, we have $(k-1)$ 0's in T. If our run takes up the entire string S, then T is just 0's. Then S and $T = I(S)$ are both winning strings in their respective games. If our run ends, then it alternates to the opposite digit at both ends. Thus, we must have 1's flanking our corresponding 0's in $I(S)$. When we eliminate the run in S to create a binary $(n-k)$-string S', we bring flanking digits together to create another run. Thus, removing the $(k-1)$ 0's in $I(S)$ in the indexing game will reduce $T = I(S)$ to $I(S')$. If S is a winning string, then some sequence of removals takes a S to a single run of identical digits. Then some sequence of removals takes $I(S)$ to a single run of two or more 0's and the converse is also true. $\square$

Example. The winning string $S = 11101100$ can be reduced to null by the steps

$$1110\{11\}00 \mapsto 111\{000\} \mapsto \{111\} \mapsto \emptyset.$$

In each step above the digits within brackets are simply removed. The transform $I(S) = 00110101$ can be reduced analogously with the steps

$$001\{101\}01 \mapsto 00\{1001\} \mapsto \{000\} \mapsto \emptyset.$$

In the above the brackets are replaced with a single 0 until we have a run of 0's.

Notice that an indexing string composed of a single run of 1's and a single run of 0's is always a losing string. We now present a very easy condition for telling when S is a losing string by looking at $I(S)$.

Proposition 8. *Let S be a non-trivial binary n-string. Then $I(S)$ is a binary n-string with an even number of ones $2m$ $0 \leq m < n/2$. S is a losing string if and only if there is a run of* 1*'s in $I(S)$ strictly greater than m.*

Proof. ($\Leftarrow$) If a string has a run of $m+1$ or more 1's we call this a main run. So assume $I(S)$ has a main run. Notice that there are at most $(m-1)$ 1's not in the main run. Each elimination step that removes a 1 from the main run also removes a 1 that is not in the main run. Since there are more 1's in the main run than 1's outside the main run, we cannot eliminate the main run and leave a run of two or more 0's.

($\Rightarrow$) We prove this by induction on n, the length of our indexing string. Thus assume any indexing string of length $j < n$ without a main run is a winning string. Say an n-string $I(S)$ has no run of $m+1$ or more 1's where $2m$ is the number of 1's in $I(S)$. We wish to prove that $I(S)$ is a winning string. Find a run q of 1's in $I(S)$ that has the maximal number of 1's. Reduce $I(S)$ to $I(S')$ by removing a 1 from this maximal run and a 1 from some other run. Then $I(S')$ has $(2m-2)$ 1's.

Suppose (for the sake of contradiction) $I(S')$ has a run s' of 1's of length $m-1+1=m$ or greater so that $I(S')$ is a losing string. Then we did not remove from a run s in $I(S)$ to produce s'. Otherwise, s would have $m+1$ or more 1's. Then s has m 1's (it cannot have more), must be a maximal run, and must be different from q which we now know must have m 1's. Then there are only two runs because there are only $2m$ 1's. But when we removed from our maximal run, we must have removed from s as well which gives our contradiction. $\square$

Thus given an indexing string with $2m$ 1's, we know that it is a losing string just by looking at the configuration of its 1's. It loses if it has a run of $m+1$ or more 1's and wins if it has no such run. Also notice that a losing string cannot have more than one main run because it only has $2m$ 1's.

Any losing indexing string must have at least two 1's. Thus, we can choose our losing indexing n-strings by picking an even number $2m$, $0 < m < n/2$, of 1's, making sure we have at least $m+1$ gathered together in a group and throwing the rest of the 1's anywhere else in the string. This group of $m+1$ or more 1's we once again call the main run. We now present our main theorem.

Theorem 9. *The number of non-trivial losing strings of length n in the same game is $2nF_{n-2}$ for all n.*

Proof. We define an **oriented losing string** to be an indexing losing string with $2m$ 1's such that the first $m+1$ digits are all 1's and the last digit is a 0. That is, we place the main run at the beginning of the string. Let T be an oriented losing string. Then $T = I(S)$ for some losing string S in the same game. Notice that $T = I(cS)$ and $rT = I(rS)$. Thus, we use $T = I(S)$ as the unique representative of the orbit of S under G. There is a way to count these oriented losing strings that gives the Fibonacci recurrence.

Because we exclude the trivial strings, we exclude the case where an even indexing string is entirely composed of 1's. Consider all the oriented losing indexing n-strings where we have a 1 placed two slots to the right of our main run (there is a 0 separating this 1 from the end of our main group). We claim that the number of losing strings of this form is equal to the number of losing strings of length $n-2$. The one-to-one correspondence is given by removing the 1 that is two slots to the right of our main run and removing a 1 in our main run to produce a losing $(n-2)$-string. By removing a 1 both from outside and inside the main run, we ensure that the resulting $(n-2)$-string is still a losing string (that it has a main run of requisite length). The inverse of this is taking an $(n-2)$-losing string, inserting a 1 to the main run and inserting a 1 two slots to the right of the main run. The above is illustrated with the following corresponding strings,

$$1111\{1\}0\{1\}0110 \leftrightarrow 111100110.$$

The first is a losing oriented indexing 11-string, the second is its corresponding losing oriented indexing 9-string. We remove or add the bracketed digits depending on the direction of the correspondence.

It is trivial to note that the number of losing indexing n-strings with a 0 placed two slots the right of the main group is the number of losing $(n-1)$-strings. We simply add or remove that 0. Thus we have the recurrence. Notice that the number of non-trivial oriented indexing losing 2-strings is $F_0 = 0$ and there is only $F_1 = 1$ oriented indexing losing 3-string: 110. Thus we have our theorem. □

References

[1] Sloane, N. J. A. The On-line Encyclopedia of Integer Sequences, published electronically at `http://www.research.att.com/~njas/sequences/`, 2005.

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