A063752 - Numbers n such that cototient(n) is square (File, Rev.4 (1))
Subfamilies and subsequences of terms

Given: cototient(n) = n – phi(n)

In this file, it will be shown that different families of integers belong to the sequence A063752.

Remark: if a(n) = p_1^r_1...p_i^r_i..... p_m^r_m is a term of A063752, then,
for each i=1,2,...,m, the number $p_i^2 * a(n)$ is also in A063752.

Moreover, if a(n) = p_1^r_1...p_i^r_i..... p_m^r_m is a term of a subsequence of A063752, then, for each i=1,2,...,m, the integer $p_i^2 * a(n)$ is also a term of the same subsequence, except for the particular subsequence of even perfect numbers, see 2.3.

There are "primitive" terms called α(n) = p_1^s_1...p_i^s_i..... p_m^s_m, with s_1, s_2,.., s_i,...,s_m = 1 or 2. These "primitive" terms generate an entire subsequence, or family, when multiply by p_i^2.

Definitions and Examples

The new non primitive terms which appear in each general sequence will be colored in green.

I) If a(n) has only one prime factor

In this case, there is only one subsequence, b(n). This subsequence contains exactly the prime powers p^{2k+1} where p is prime. They form the sequence A246551.
If **b(n) = p^{2k+1}** , then **cototient(b(n)) = $(p^k)^2$**.
In particular, if p is prime, cototient(p) = 1. The primes in this case are exactly the primitive terms : **(β(n)) = p** of this subsequence with **cototient(β(n)) = 1.**

Examples:
cototient(5) = 1, and 125 = 5^3, cototient(125) = 5^2.

II) If a(n) has two distinct prime factors

These terms form the sequence A323916. The first few terms are {6, 21, 24, 28, 54, 68, 69, 96, 112, 124, 133, 141, 189, 216,...}. There are two subsequences in this case which are a partition of A323916 and also a third particular subsequence.

The three subsequences of A063752 are described in the following section where they are defined from their primitive representation.

2.1) Define the first case by **γ(n) = p * q**, p < q with **cototient(γ(n))** = **p+q-1 = M^2** is square.
The first primitive terms are: {6, 21, 69, 133, 141, 237, 301, 481, 501, 589, 669, 781,...}.

Couples (p,q) with p < q can be found by solving the Diophantine equation:
$p+q-1 = M^2$, with p, q primes.
Discussion: for M=2, (p=2, q=3) is solution and there is not other solution for M even >= 4. Otherwise, for M odd >= 3, following Goldbach's conjecture, every $M^2 + 1$ is even >= 10 and so can be expressed as the sum of two primes p+q. The number of such decompositions, so the number of couples (p,q) which are solutions of this Diophantine equation, is in A002375.

M = 2, p+q = 5, (p,q) = (2,3) with 2+3-1 = 2^2, only one solution,
M = 3, p+q = 10, (p,q) = (3,7) with 3+7-1 = 3^2, only one solution too,
M = 5, p+q = 26, (p,q) = (3,23), (7,19), so two solutions,
M = 7, p+q = 50, (p,q) = (3,47), (7,43), (13,37), (19,31), four solutions.

Some values for (γ(n),p,q,M) = (6,2,3,2), (21,3,7,3), (69,3,23,5), (133,7,19,5), (141,3,47,9), (237,3,79,9), (301,7,43,7), (481,13,37,7),...

The general terms are **c(n)** = $\mathbf{p^{2s+1} * q^{2t+1}}$ with p < q, r,s >= 0, **cototient(c(n)) =** $\mathbf{(p^s * q^t * M)^2}$.
The first few terms are: {6, 21, 24, 54, 69, 96, 133, 141, 189, 216, 237, 301, 384, 481,..}.
These terms form the sequence A323917.

Examples:
69 = 3 * 23, 3+23-1= 5^2, so cototient(69) = 5^2.
96 = 2^5 * 3, 2+3-1= 2^2 and cototient(96) = $(2^2 * 3^0 * 2)^2 = 8^2$.

2.2) Define the second by **δ(n) =** $\mathbf{p^2 * q}$ with **cototient(δ(n)) = p * (p+q-1) =** $\mathbf{M^2}$**.**
The first primitive terms are: {28, 68, 124, 284, 388, 508, 657, 796, 964, 1025,...}.
Some values of (δ(n),p,q,M): (28,2,7,4), (68,2,17,6), (124,4,31,8), (284,2,71,12), (657,3,73,15),...

The general terms **d(n) =** $\mathbf{p^{2s} * q^{2t+1}}$ with s>=1, t>=0 verify that:
cototient(d(n)) = $\mathbf{(p^{s-1} * q^t * M)^2}$. The first few terms are: {28, 68, 112, 124, 272, 284, 388, 448, 496, 508, 657, 796, 964, 1025,...}. These terms form the sequence A323918.

Some particular cases:

If p = 2, then d(n) = $2^{2s} * q^{2t+1} = 4^s * q^{2t+1}$ with 2*(q+1) is square, so q belongs to A066436, and for “primitive” terms, δ(n) = 2^2 * q = 4*q with always 2*(q+1) square.
The first terms for p=2 are: {28, 68, 112, 124, 272, 284,388,...}.
Examples:
28 = 2^2 * 7, 2*(7+1) = 4^2 and cototient(28) = 4^2.
272 = 2^4 * 17, 2*(17+1) = 6^2 and cototient(272) = $(2^1 * 6)^2 = 12^2$.

If p = 3, then d(n) = $3^{2s} * q^{2t+1} = 9^s * q^{2t+1}$ with 3*(q+2) is square, so q in A201715.
Example, 657 = 3^2 * 73, 3*(73+2) = 15^2 and cototient(657) = 15^2.

If p = 5, then d(n) = $5^{2s} * q^{2t+1} = 25^s * q^{2t+1}$ with 5*(q+4) is square, so q in A201786.
Example 1025 = 5^2 * 41, 5*(41+4) = 15^2 and cototient(1025) = 15^2.

2.3) **Even perfect numbers** (A000396). This subsequence is a special supplementary subsequence. By Euclid-Euler theorem, a perfect number has the form:
$2^{p-1} * (2^p – 1)$ where $M_p = 2^p – 1$ is a Mersenne prime (see A000043 - A000668).
For this case, **cototient**$\mathbf{(2^{p-1} * (2^p – 1)) = (2^{p-1})^2}$

The first even perfect number, 6, belongs to the second subsequence A323917 while the other ones belong to the third subsequence A323918.

Examples: 6 = 2 * 3 and cototient(6) = 2^2 and for,
8128 = 2^6 * 127 then cototient(8128) = $(2^6)^2 = 64^2$ = 4096.

III) If a(n) has three distinct prime factors.

Some brief remarks about these integers with three distinct prime factors which form the OEIS sequence A306670. There are three subsequences in this case.
The first term with three prime distinct factors in the sequence A306670 is 345 = 3 * 5 * 23, the third one is 468 = $2^2 * 3^2 * 13$, the twenty-sixth is: 7105 = $5 * 7^2 * 29$.

3.1) Define this first case by **ε(n) = p * q * r**, and, **p*q*r – (p-1)*(q-1)*(r-1) = M^2**,
cototient(ε(n)) = p*q*r – (p-1)*(q-1)*(r-1) = M^2.
The first primitive terms are: {345, 465, 1545, 1833, 2737, 2769, 3145, 3585, 3657, 3945,..}.

The general terms are **e(n) = $p^{2s+1} * q^{2t+1} * r^{2u+1}$** with s,t>=0, u>=1 and p,q,r primes such that: **cototient(l(n)) = $(p^s * q^t * r^u * M)^2$**.
The first few terms are: {345, 465, 1545, 1833, 2737, 2769, 3105, 3145, 3585, 3657, 3945,...}.

Example: 345 = 3*5*23 and cototient(345) = 3*5*23 – 2*4*22 = 169 = 13^2.

3.2) Define the second case by **λ(n) = p^2 * q * r**, and **p*[p*q*r – (p-1)*(q-1)*(r-1)] = M^2,**
cototient(λ(n)) = p * [p*q*r – (p-1)*(q-1)*(r-1)] = M^2.
The first primitive terms are: {7105, 14841, 24321, 40033, 42129, 55521, 56425, 58825,...}.

The general terms are **l(n) = $p^{2s} * q^{2t+1} * r^{2u+1}$** with s>=1, t,u>=0 and p,q,r primes such that: **cototient(l(n)) = $(p^{s-1} * q^t * r^u * M)^2$**.

Example: 7105 = $5 * 7^2 * 29$ and cototient(7105) = 7*(5*7*29 – 4*6*28) = 49^2.

3.3) Define the third case by **ζ(n) = $p^2 * q^2$ * r** with
cototient(ζ(n)) = p * q * [p*q*r – (p-1)*(q-1)*(r-1)] = M^2.
The first primitive terms are: {468, 1332, 2628, 4100, 6516, 8428, 12132,...}.

The general terms are **z(n) = $p^{2s} * q^{2t} * r^{2u+1}$** with s,t>= 1, u>=0 and, p,q,r primes such that **cototient(z(n)) = $(p^{s-1} * q^{t-1} * r^u * M)^2$.**
The first few terms are: {468, 1332, 1872, 2628, 4100, 4212, 5328, 6516, 7488, 8428,...}.

Example: 468 = $2^2 * 3^2 * 13$ and cototient(468) = 2*3*(2*3*13 – 1*2*12) = 18^2.

IV) If a(n) has four distinct prime factors.

There is a new sequence {420, 1680, 2340, 2436, 3300,...} and other subsequences with four distinct prime factors that can be found with similar conditional requirements as displayed here.

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Mar 05 2019