UNIT DISTANCE GRAPHS OF ORDER 7

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Abstract. The connected unit distance graphs on up to 7 vertices are determined.

We say that a graph G with vertices V and edges $E \subseteq \{\{v, w\} : v, w \in V \text{ with } v \neq w\}$ is a *unit-distance graph* when there exists an injection $f : V \to \mathbf{R}^2$ such that for every $\{v, w\} \in E$, $|f(v) - f(w)| = 1$. We say that a graph is a *forbidden subgraph* if it is not a unit-distance graph, since then it may not appear as a subgraph of any unit-distance graph. We will show that the following are forbidden subgraphs:

$$\begin{aligned}
K_4 &= [(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)]\\
K_{2,3} &= [(0,3),(0,4),(1,3),(1,4),(2,3),(2,4)]\\
F_1 &= [(0,1),(0,2),(0,3),(0,4),(1,2),(1,5),(2,3),(3,4),(4,5)]\\
F_2 &= [(0,1),(0,2),(0,6),(1,2),(1,3),(1,5),(2,3),(3,4),(4,5),(4,6),(5,6)]\\
F_3 &= [(0,1),(0,2),(0,3),(0,6),(1,2),(1,6),(2,3),(2,4),(3,4),(4,5),(5,6)]\\
F_4 &= [(0,1),(0,3),(0,4),(0,6),(1,2),(2,3),(2,5),(3,4),(4,5),(5,6)]
\end{aligned}$$

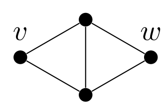
The notation above is a list of edges between integer-labeled nodes.

We will in fact show the following.

Proposition. *A graph on $n \leq 7$ vertices is a unit-distance graph if and only if it does not contain an isomorphic copy of K_4, $K_{2,3}$, F_1, F_2, F_3, or F_4 as a subgraph.*

Proof. We first show these are indeed forbidden subgraphs.

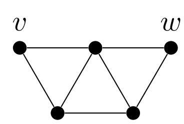
K_4 contains the following unit-distance subgraph:



and requires v and w to be adjacent. However, the graph above can only be realized as a unit-distance graph with v and w at distance $\sqrt{3}$ from each other, so K_4 is a forbidden subgraph.

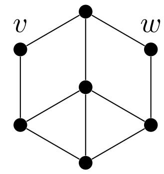
$K_{2,3}$ is a forbidden subgraph since any distinct circles of radius 1 intersect in at most 2 points.

F_1 contains the following unit-distance subgraph:



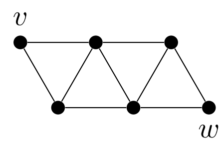
and requires an additional common neighbor of v and w. However, the graph above can only be realized as a unit-distance graph with v and w at distance 2 from each other, so they cannot have an additional common unit-distance neighbor. So, F_1 is a forbidden subgraph.

F_2 contains the following unit-distance subgraph:



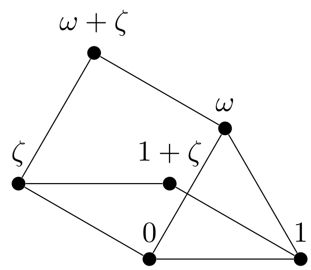
and requires v and w to be adjacent. While the graph above does have some flexibility in how it is embedded, every 4-cycle in a unit-distance graph must be a rhombus, which forces v and w above to be at distance $\sqrt{3}$ from each other. So, F_2 is a forbidden subgraph.

F_3 contains the following unit-distance subgraph:



and requires a common neighbor of v and w. However, the graph above can only be realized as a unit-distance graph with v and w at distance greater than 2 from each other, so they cannot have a common unit-distance neighbor. So, F_3 is a forbidden subgraph.

F_4 contains the following unit-distance subgraph, where any embedding can be translated and rotated to



with $\omega = \exp(i\pi/3)$ and $\zeta \in \mathbf{C}$ with $|\zeta| = 1$. Even though ζ is arbitrary, note

$$|(\omega + \zeta) - (1 + \zeta)| = |\omega - 1| = 1.$$

F_4 requires an additional common neighbor of $\omega + \zeta$ and 1, but both unit-distance candidates, namely ω and $1 + \zeta$, are occupied. So, F_4 is a forbidden subgraph.

With the help of SAGE, one can generate the 222 non-isomorphic connected graphs on 7 vertices that do not contain the forbidden subgraphs above. Each is isomorphic to a unit-distance graph with explicit coordinates given here:

`http://hansparshall.com/txt/n7UDcoordinates.txt`

where each graph is expressed as a list of edges between complex numbers of distance 1 apart. To arrive at these coordinates, we considered $\omega_1 = \exp(i\pi/3)$ and $\omega_3 = \exp(i \arccos(5/6))$ and the set

$$V = \{\omega_1^a \omega_3^b : a \in \{0, 1, 2, 3, 4, 5\}, b \in \{0, 1\}\}.$$

Then each explicit coordinate above is a member of the sumset $V + V = \{v + w : v, w \in V\}$. $\square$