A Note on Modular Partitions and Necklaces

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Abstract

Following Jens Voß, let $T(n,k)$ be the number of "modular partitions" of n into k parts, that is, the number of k-tuples $(u_1, u_2, \ldots, u_k)$ with $0 \le u_1 \le u_2 \le \cdots \le u_k \le n-1$ such that $\sum_j u_j \equiv 0 \bmod n$. The purpose of this note is to use Molien's theorem to show that $T(n,k)$ is also equal to the number of bi-colored necklaces with n beads of one color and k beads of another color.

1 Introduction

Following Jens Voß [9], let $T(n,k)$ be the number of k-tuples $u = (u_1, u_2, \ldots, u_k)$ with $0 \le u_1 \le u_2 \le \cdots \le u_k \le n-1$ such that $\sum_j u_j \equiv 0 \bmod n$. Stated another way, $T(n,k)$ is the number of ways to write 0 as a sum of k elements of $\mathbb{Z}/n\mathbb{Z}$. Voß calls u a *modular partition* of n into k parts. He computed the numbers $T(n,k)$ for $n+k \le 20$, and part of his table is shown here (the rows correspond to $n = 0, 1, 2, \ldots, 10$ and the columns to $k = 0, 1, 2, \ldots$):

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1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,...
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,...
1,1,2,2,3,3,4,4,5,5,6,6,7,7,8,8,9,9,10,...
1,1,2,4,5,7,10,12,15,19,22,26,31,35,40,46,51,57,...
1,1,3,5,10,14,22,30,43,55,73,91,116,140,172,204,245,...
1,1,3,7,14,26,42,66,99,143,201,273,364,476,612,776,...
1,1,4,10,22,42,80,132,217,335,504,728,1038,1428,1944,...
1,1,4,12,30,66,132,246,429,715,1144,1768,2652,3876,...
1,1,5,15,43,99,217,429,810,1430,2438,3978,6310,...
1,1,5,19,55,143,335,715,1430,2704,4862,8398,...
1,1,6,22,73,201,504,1144,2438,4862,9252,...
...
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The OEIS [5] now contains three versions of this table: see entries A047996, A241926, and A037306. The table *appears* to be symmetric about the main diagonal, although this is not obvious from the definition. For example, the entry $T(3,5) = 7$ corresponds to these seven u-vectors:

$$00000,\ 00012,\ 00111,\ 00222,\ 01122,\ 11112,\ 12222\,. \tag{1}$$

These vectors have five components, entries 0, 1, and 2, and a sum which is a multiple of 3. On the other hand, the entry $T(5,3) = 7$ corresponds to these seven u-vectors:

$$000,\ 014,\ 023,\ 113,\ 122,\ 244,\ 334\,. \tag{2}$$

These have three components, entries from 0 to 4, and a sum which is a multiple of 5. There is no obvious bijection between the two lists.

Consulting the OEIS, it appears that the nth row of the table gives (a) the coefficients of the Molien series for the regular representation of the cyclic group of order n, and also (b) the numbers of inequivalent necklaces with $n+k$ beads, n of one color and k of a another color. (Rows 2, 3, ... appear to match OEIS entries A008619, A007997, A008610, A008646, A032191, A032192, A032193, etc.) It is the purpose of this note to use Molien's theorem to prove that these empirical observations are in fact correct.

2 Molien's theorem

Molien's theorem states that if G is a finite group of complex $n \times n$ matrices, and a_d ($d = 0, 1, \ldots$) is the number of linearly independent homogeneous polynomials of degree d that are invariant under the action of G, then

$$\Phi_G(\lambda) \;=\; \sum_{d=0}^{\infty} a_d \lambda^d \;=\; \frac{1}{|G|} \sum_{M \in G} \frac{1}{\det(I - \lambda M)} \tag{3}$$

(see for example [4]; [2, p. 77], [7, p. 87], [8, p. 29]). $\Phi_G(\lambda)$ is called the *Molien series* for G.

We apply the theorem to two different (but equivalent) matrix groups, G_1 and G_2, both abstractly isomorphic to the cyclic group $\mathcal{C}_n$ of order n. The first, G_1, is the group of $n \times n$ permutation matrices generated by the cyclic permutation $\sigma := (0, 1, 2, \ldots, n-1)$. For example, if $n = 4$, G_1 consists of the four powers of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} . \tag{4}$$

The second group, G_2, is the group of $n \times n$ diagonal matrices generated by the matrix

$$\operatorname{diag}\{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\} , \tag{5}$$

where $\zeta = e^{2\pi i/n}$. For $n = 4$, the generator is $\operatorname{diag}\{1, i, -1, -i\}$.

Since G_2 is in fact the result of diagonalizing the elements of G_1, corresponding elements of the two groups have the same eigenvalues, and G_1 and G_2 have the same Molien series.

We will show that the invariants of G_1 of degree k correspond to necklaces of $n+k$ beads, n of one color and k of another color, and that the invariants of G_2 of degree k correspond to ways to write 0 as a sum of k terms in $\mathbb{Z}/n\mathbb{Z}$. Evaluating the Molien series for G_1 gives a familiar formula (see (8)) for the number of such necklaces, and since the two Molien series are the same, this is also the number of modular partitions $T(n, k)$. The results are summarized in Theorem 1. The fact that $T(n, k) = T(k, n)$ is then an immediate corollary.

3 The group G_1 and its invariants

The group G_1 is the regular permutation representation of the cyclic group $\mathcal{C}_n$. It is well-known that for each d dividing n, $\mathcal{C}_n$ contains $\phi(d)$ elements which are a product of n/d

cycles of length d, where $\phi(d)$ is the Euler totient function (A000010) (see for example [1]). Each such element contributes a term $(1-\lambda^d)^{-n/d}$ to the Molien series, so

$$\Phi_{G_1}(\lambda) \; = \; \frac{1}{n} \sum_{d|n} \phi(d) \frac{1}{(1-\lambda^d)^{n/d}} \,. \tag{6}$$

We wish to determine a_k, the coefficient of λ^k in this expression. The contribution to the sum from the d term will be zero unless d also divides k, so we may restrict the sum to values of d that divide the greatest common divisor of n and k. Using the binomial theorem, we obtain

$$a_k \; = \; \frac{1}{n} \sum_{d|\gcd(n,k)} \phi(d) \binom{\frac{n}{d}+\frac{k}{d}-1}{\frac{k}{d}} \,. \tag{7}$$

This may be rewritten more symmetrically as

$$a_k \; := \; \frac{1}{n+k} \sum_{d|\gcd(n,k)} \phi(d) \binom{\frac{n+k}{d}}{\frac{k}{d}} \,, \tag{8}$$

and now it is obvious that interchanging the roles of n and k leaves the number unchanged (as it must, since we can obviously exchange the names of the colors of the two kinds of beads without affecting the number of necklaces). This is a classical formula (Lucas, [3, pp. 501–503], Riordan, [6, p. 162]). It can also be easily derived using Pólya's theorem (cf. [1]).

Next we consider the invariants themselves and show how they correspond to necklaces. Let the variables on which G_1 acts be labeled $x_0, x_1, \ldots, x_{n-1}$. If we take an arbitrary monomial of degree k, say

$$x_0^{\ell_0} x_1^{\ell_1} \cdots x_{n-1}^{\ell_{n-1}}, \text{ with } \sum_j \ell_j = k \,, \tag{9}$$

and form the sum of its images under all cyclic shifts of the subscripts, the result is invariant under the action of G_1, and conversely, every invariant can be decomposed into a sum of such invariants.

We describe necklaces with $n+k$ beads of two colors by binary vectors of length $n+k$, containing n 0's and k 1's, with the understanding that cyclic shifts of the vector correspond to the same necklace. For example, there are five necklaces with four beads of one color and three of the other color, which are described by the binary vectors

$$0000111,\; 0001011,\; 0001101,\; 0010011,\; 0010101\,. \tag{10}$$

The correspondence between invariants of degree k and necklaces with $n+k$ beads is that a monomial (9) corresponds to the necklace with binary vector

$$0^{\ell_0} 1\, 0^{\ell_1} 1\, 0^{\ell_2} 1 \cdots 1\, 0^{\ell_{n-1}} 1 \,, \tag{11}$$

where there are n 0's and k 1's. In other words, the exponents in the monomial (9) specify the lengths of the successive strings of beads of one color. Choosing a different monomial term from an invariant just gives a different cyclic shift of the necklace. The invariants corresponding to the necklaces in (10) are respectively

$$x_0^4+x_1^4+x_2^4,\; x_0^3x_1+x_1^3x_2+x_2^3x_0,\; x_0^3x_2+x_1^3x_0+x_2^3x_1,\; x_0^2x_1^2+x_1^2x_2^2+x_2^2x_0^2,\; x_0^2x_1x_2+x_1^2x_2x_0+x_2^2x_0x_1 \,.$$

4 The group G_2 and its invariants

The eigenvalues of the rth power of the generator (5) are $\zeta^{jr}, j = 0, 1, \ldots, n-1$, so the Molien series for G_2 is

$$\Phi_{G_2} = \frac{1}{n} \sum_{r=0}^{n-1} \frac{1}{\prod_{j=0}^{n-1}(1-\lambda\zeta^{jr})} . \quad (12)$$

Since G_2 consists of diagonal matrices, any monomial term in an invariant must itself be invariant. If (9) is an invariant of degree k, then we have

$$\sum_{j=0}^{n-1} \ell_j = k, \quad \sum_{j=0}^{n-1} j\ell_j \equiv 0 \bmod n , \quad (13)$$

which is the definition of a modular partition of n into k parts, the parts $u_1, u_2, \ldots, u_k$ being 0 (ℓ_0 times), 1 (ℓ_1 times), ..., $n-1$ (ℓ_{n-1}) times. So by Molien's theorem,[1]

$$\Phi_{G_2}(\lambda) = \sum_{k=0}^{\infty} T(n,k)\lambda^k .$$

Since $\Phi_{G_1}(\lambda) = \Phi_{G_2}(\lambda)$, we have proved:

Theorem 1 *The following are equal:*
(i) $T(n,k)$, the number of ways of writing 0 as a sum of k terms in $\mathbb{Z}/n\mathbb{Z}$,
(ii) the number of bi-colored necklaces with n beads of one color and k beads of another color,
(iii)

$$\frac{1}{n+k} \sum_{d|\gcd(n,k)} \phi(d) \binom{\frac{n+k}{d}}{\frac{k}{d}} . \quad (14)$$

Corollary 2 $T(n,k) = T(k,n)$.

References

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[1] One can also show directly, by purely combinatorial arguments, that the right-hand side of (12) is a generating function for the numbers $T(n,k)$ for fixed n.

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