The Number of Square-Free Self-Avoiding Walks in 2-dimensions.

Anne E. Edlin

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Abstract

A walk is self-avoiding if it never meets itself. A walk can be viewed as a word if each step is assigned a 'letter'. A walk is square-free if its word contains no subwords of the form xx. This paper proves that the number of square-free self-avoiding walks in 2-dimensions of any length is finite.

The maple packages used to discover this result are available at this papers web site **http://www.math.temple.edu/~anne/sqfrwalk.html**.

Preface

Throughout this paper walks are viewed as words, written in the form $[w_1, w_2, \ldots, w_n]$, where each letter, w_i represents one step of the walk. In 2-dimensions we can use the alphabet $\{1, -1, 2, -2\}$ as our set of possible steps. Here 1 represents a step to the right, -1 a step to the left, 2 a step up and -2 a step down.

We define a factor of a word in the following way. Given a word $w = [w_1, w_2, \ldots, w_n]$, a *factor* of w is any subword of the form $[w_k, w_{k+1}, \ldots, w_{k+j}]$ where $1 \leq k \leq n$ and $0 \leq j \leq n-1$.

A *self-avoiding walk* in 2-dimensions is a path on the 2-dimensional lattice that does not visit the same site twice [1]. Using our notation this is equivalent to a word is self-avoiding if it contains no factors for which the number of 1s and -1s are equal and the number of 2s and -2s are equal. My Maple package walk (available from **http://www.math.temple.edu/~anne/sqfrwalk.html**) can be used to derive or count the number of self-avoiding walks that avoid an input set of mistakes in any given dimension.

A word is *square-free* if it contains no factors of the form xx, where x is any word. My Maple package Squares (available from **http://www.math.temple.edu/~anne/sqfrwalk.html**) can be used to derive square words on any given alphabet up to the required length.

The Sequence

The number of square-free self-avoiding walks for n from 0 to 20 are:

[1, 4, 8, 16, 16, 16, 16, 16, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].

A Non-Computer Proof

By making our walks self-avoiding we know we must eliminate all immediate back steps and all polygons, at the very least. This means that none of the following set of words may appear as a factor of any of our words:

$$\{ \quad [1,-1],[2,-2],[-1,1],[-2,2],[1,2,-1,-2],[2,-1,-2,1],[-1,-2,1,2], [-2,1,2,-1],[-1,2,1,-2],[-2,-1,2,1],[1,-2,-1,2],[2,1,-2,-1] \quad \}.$$

The fact the walks are also square free eliminates double steps and double 'corners', that is paths like (right, up, right, up). So we must also eliminate all of the following as factors:

$$\{ \quad [1,1],[-1,-1],[2,2],[-2,-2],[-1,1,-1,1],[2,1,2,1],[-2,1,-2,1], [1,-1,1,-1],[2,-1,2,-1],[-2,-1,-2,-1],[1,2,1,2],[-1,2,-1,2], [-2,2,-2,2],[1,-2,1,-2],[-1,-2,-1,-2],[2,-2,2,-2] \quad \}.$$

So let us now try to form a square-free self-avoiding walk. By symmetry it does not matter in which direction we start, so let our first step be a 1.

Now our second step may not be -1 as the walk is self-avoiding, and it can not be 1, because our walk is square-free. So our next step must be 2 or -2. Again by symmetry it does not matter which we chose, so we will pick 2.

Our walk so far is $[1,2]$. Now as before we may not pick -2 or 2, because our previous step was 2, so we must pick 1 or -1. Both cases are very similar so I will only look at the case that the next step is 1 in this paper. The case when the next step is -1 is left to the reader.

We now have $[1,2,1]$. For our next step I may not pick -1 or 1, because the last step was a 1, and I may not pick 2, because $[1,2,1,2]$ is a square (of $[1,2]$), this means I am forced to pick a -2.

Now we have $[1,2,1,-2]$. From here I may not pick -2 or 2 as usual, and I may not pick -1, or the last four steps will form a polygon $[2,1,-2,-1]$, and so will not be self-avoiding. Thus I am forced to pick 1.

I am forced into my next step up until the eighth step. Here is the position after 7 steps:

$[1,2,1,-2,1,2,1]$. See Figure 1.

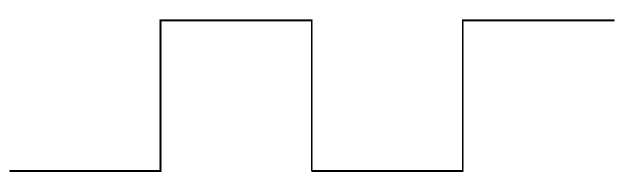

Figure 1: A seven step square free self-avoiding walk

Now based on the previous analysis I must chose -2 as my next step, but if I do this I will have the $[1,2,1,-2]$ twice in succession. So as I want my walk to be square-free I am stuck, and can take no further steps. Thus for $n \geq 8$ the number of square-free self-avoiding walks is zero.

References

[1] Neal Madras and Gordon Slade (1996). *The Self-Avoiding Walk*. Birkhauser, Boston.