An identity for A039619

DAVID CALLAN

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Proposition. *For $n \geq 2$,*

$$(n-1)! \sum_{k=1}^{n-1} \binom{2n-1-k}{n-1-k} \frac{1}{k} = \frac{(2n-1)!}{n!} \left(H_{2n-1} - H_n \right)$$

Proof (bijective). For a list of positive integers, say an entry a is *left-full* if $1, 2, \ldots, a-1$ all occur in the list and before a. Thus, the left-full entries in 24173 are 1 and 3. Clearly, if 1 occurs in the list it is always left-full, while if 1 does not occur then no entry is left-full.

Let $\mathcal{A}_n$ denote the set of lists of $n-1$ distinct elements of $[2n-1]$ with one marked left-full entry (and possibly other left-full entries). For example, with a dot denoting the marked entry, $\mathcal{A}_3 = \{\dot{1}2,\ 1\dot{2},\ \dot{1}3,\ \dot{1}4,\ \dot{1}5,\ 2\dot{1},\ 3\dot{1},\ 4\dot{1},\ 5\dot{1}\}$.

To count $\mathcal{A}_n$ by the location i $(1 \leq i \leq n-1)$ of the marked left-full entry, let us construct its members entrywise left to right. With the falling factorial notation $n^{\underline{i}} := n(n-1)\cdots(n-i+1)$, there are $(2n-1)^{\underline{i-1}}$ choices for the first $i-1$ entries since the only restriction is that they must be distinct, then just 1 choice for the ith entry since it must be the smallest unused element of $[2n-1]$, and then $(2n-i-1)^{\underline{n-i}}$ choices for the remaining entries. Thus

$$\begin{aligned}
|\,\mathcal{A}_n\,| &= \sum_{i=1}^{n-1} (2n-1)^{\underline{i-1}} (2n-i-1)^{\underline{n-i}} \\
&= (2n-1)^{\underline{n-1}} \sum_{i=1}^{n-1} \frac{1}{2n-i} \\
&= \frac{(2n-1)!}{n!} \left(\frac{1}{2n-1} + \frac{1}{2n-2} + \cdots + \frac{1}{n+1} \right) \\
&= \frac{(2n-1)!}{n!} (H_{2n-1} - H_n)
\end{aligned}$$

Now let us count $\mathcal{A}_n$ by the *value* of the marked entry. To construct the members of $\mathcal{A}_n$ for which k is the marked left-full entry, arrange $1, 2, \ldots, k-1$ to the left of k thus: $a_1\ a_2\ \ldots\ a_{k-1}\ k$, for $(k-1)!$ choices, then choose $n-1-k$ elements from $[k+1, 2n-1]$ to form the remainder of the list—$\binom{2n-1-k}{n-1-k}$ choices—and lastly, arrange these $n-1-k$ elements together with k dots representing the already placed entries—$\frac{(n-1)!}{k!}$ choices.

Thus

$$
\begin{aligned}
|\,\mathcal{A}_n\,| &= \sum_{k=1}^{n-1}(k-1)!\binom{2n-1-k}{n-1-k}\frac{(n-1)!}{k!} \\
&= (n-1)!\sum_{k=1}^{n-1}\binom{2n-1-k}{n-1-k}\frac{1}{k}
\end{aligned}
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□

Department of Statistics, University of Wisconsin-Madison
1300 University Ave, Madison, WI 53706-1532
callan@stat.wisc.edu