Enumerative Geometry

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Given a complex projective variety $\mathbb{V}$ (as defined in [1]), we wish to count the curves in $\mathbb{V}$ that satisfy certain prescribed conditions. Let $\widetilde{\mathbb{C}^n}$ denote complex projective n-dimensional space. In our first example, $\mathbb{V}=\widetilde{\mathbb{C}^2}$, the complex projective plane; in the second and third, $\mathbb{V}$ is a general hypersurface in $\widetilde{\mathbb{C}^n}$ of degree $2n-3$. Call such $\mathbb{V}$ a **cubic twofold** when $n=3$ and a **quintic threefold** when $n=4$.

Our interest is in **rational curves**, which include all lines (degree 1), conics (degree 2) and singular cubics (degree 3). No elliptic curves are rational. The word "rational" here refers to the affine parametrization of the curve – a ratio of polynomials – and the curve is of degree d if the polynomials are of degree at most d. For instance, the circle $x^2+y^2=1$ is represented as

$$x=\frac{1-t^2}{1+t^2}, \qquad y=\frac{2t}{1+t^2}, \qquad -\infty<t<\infty.$$

The lemniscate of Bernoulli has degree 4 and is represented as

$$x=\frac{1-t^4}{1+6t^2+t^4}, \qquad y=\frac{2t\left(1-t^2\right)}{1+6t^2+t^4}, \qquad -\infty<t<\infty.$$

It is also defined implicitly:

$$\left(x^2+y^2\right)^2=x^2-y^2$$

and clearly possesses a singularity (vanishing gradient) at the origin. The semicubical parabola $y^2=x^3$ and four-petal rose

$$\left(x^2+y^2\right)^3=4x^2y^2$$

possess likewise. All rational curves, smooth or not, have genus 0.

0.1. Rational Plane Curves Passing Through Points. In the following, we use homogeneous coordinates. Given two distinct points (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) in $\widetilde{\mathbb{C}^2}$, there is exactly one line passing through both because the simultaneous system of equations

$$aX_j + bY_j + cZ_j = 0, \qquad j \in \{1, 2\}$$

has a unique solution (a, b, c) in $\widetilde{\mathbb{C}^2}$ (up to a common scalar). It is a little harder to prove the corresponding result for conics. Given five points (X_j, Y_j, Z_j) in *general position*, there is exactly one conic passing through all five via study of

$$aX_j^2 + bX_jY_j + cY_j^2 + dX_jZ_j + eY_jZ_j + fZ_j^2 = 0, \qquad j \in \{1, 2, 3, 4, 5\}$$

in $\widetilde{\mathbb{C}^5}$. Hence we have $K_1 = K_2 = 1$, where K_d is defined as the number of rational curves in $\widetilde{\mathbb{C}^2}$ of degree d passing through $3d - 1$ general points. The quantity $3d - 1$ turns out to be the critical threshold for our question: less would give an answer of infinity, more would give an answer of zero [2].

Proving that $K_3 = 12$ involves a heavy dose of algebraic geometry [3, 4]. Credit for this accomplishment (in the mid-1800s) is assigned variously to Chasles [5] and Steiner [6].

Kontsevich's famous recursion [7, 8, 9]:

$$K_d = \sum_{\substack{d_1+d_2=d,\\ d_1\geq 1, d_2\geq 1}} K_{d_1}K_{d_2}\left[d_1^2 d_2^2\binom{3d-4}{3d_1-2} - d_1^3 d_2\binom{3d-4}{3d_1-1}\right], \qquad d > 1$$

was not found until recently (in 1994). Its astonishing proof drew upon ideas not from geometry but from mathematical physics, specifically, quantum field theory and string theory. Other relevant recursions for curve counting are known [7, 10, 11, 12] but these are too complicated for us to discuss here.

The asymptotics for K_d are [11, 13]

$$\frac{K_d}{(3d-1)!} \sim \frac{(0.1380093466...)^d}{d^{7/2}}\left(\frac{6.0358078488...}{1} - \frac{2.2352424409...}{d} + \frac{0.0543137879...}{d^2} + \cdots\right)$$

as $d \to \infty$, obtained using a device due to Zagier called the "asymp$_k$ trick". No closed-form expression for these constants is known.

0.2. Lines On a Hypersurface. The fact that exactly 27 lines lie on a cubic twofold in $\widetilde{\mathbb{C}^3}$ is a well-known theorem [14, 15] due to Cayley & Salmon (in 1849). Somewhat later, Schubert proved (in 1886) that exactly 2875 lines lie on a quintic threefold in $\widetilde{\mathbb{C}^4}$. Thus we have $M_3 = 27$ and $M_4 = 2875$, where M_n is defined as the number of lines on a general hypersurface in $\widetilde{\mathbb{C}^n}$ of degree $2n-3$. Expanding on these results, van der Waerden proved (in 1933) that

$$M_n = \frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left((1-x)\prod_{k=0}^{2n-3}(2n-3-k+kx)\right)\Bigg|_{x=0}$$

and Zagier [9, 13] obtained asymptotics

$$M_n \sim \sqrt{\frac{27}{\pi}}(2n-3)^{2n-7/2}\left(1-\frac{9}{8n}-\frac{111}{640n^2}-\frac{9999}{25600n^3}+\cdots\right)$$

as $n\to\infty$. In this case, closed-form expressions are available.

0.3. Rational Curves On a Quintic Threefold. The number of conics on a cubic twofold is infinity. In contrast, the number of conics on a quintic threefold is 609250. Our discussion at this point becomes highly speculative – it is merely conjectured (by Clemens [8]) that the number n_d of degree d rational curves on a quintic threefold is finite – but the following calculations are known to be valid at least for $d\le 9$. Define $f_0(q)$, $f_1(q)$, $f_2(q)$ via power series expansion of a certain hypergeometric function [4]:

$$\sum_{d=0}^{\infty} q^d \frac{\prod_{j=1}^{5d}(5w+j)}{\prod_{k=1}^{d}(5w+k)^5} = f_0(q)+f_1(q)w+f_2(q)w^2+\cdots.$$

It follows that

$$f_0(q)=\sum_{d=0}^{\infty} q^d\frac{(5d)!}{(d!)^5}, \qquad f_1(q)=\sum_{d=0}^{\infty} q^d\left(\frac{(5d)!}{(d!)^5}\sum_{i=d+1}^{5d}\frac{1}{i}\right)$$

(a similar expression for $f_2(q)$ would be good to see). We then define rational numbers N_d recursively from

$$f_2(q)=\frac{1}{2}\frac{f_1(q)^2}{f_0(q)}+\frac{1}{5}\sum_{d=0}^{\infty} d\,N_d q^d f_0(q)\exp\left(d\,\frac{f_1(q)}{f_0(q)}\right),$$

yielding

$$\{N_d\}_{d=1}^{\infty}=\left\{2875,\frac{4876875}{8},\frac{8564575000}{27},\frac{15517926796875}{64},229305888887648,...\right\}.$$

Such numbers are examples of **Gromov-Witten invariants**, which count not only the rational curves we desire, but also capture (unwanted) additional structure [8]. The final step is another recursion [4, 16]:

$$N_d = \sum_{h\,|\,d} \frac{n_{d/h}}{h^3}$$

yielding

$$\{n_d\}_{d=1}^{\infty} = \{2875, 609250, 317206375, 242467530000, 229305888887625, ...\}\,.$$

It is, again, merely conjectured (by Gopakumar & Vafa [8, 9]) that all numbers n_d obtained in this manner are indeed integers. Much work lies ahead to rigorously confirm everything written here. The asymptotics for n_d remain open.

0.4. Addendum. Let S be a cubic twofold and let H_d be the number of rational curves on S of degree d passing through $d-1$ general points on S. Traves [9, 17] gave the values

$$\{H_d\}_{d=1}^{\infty} = \{27, 27, 72, 216, 459, 936, ...\}$$

and conjectured that H_d is always finite. A recursive formula for H_d (à la Kontsevich for K_d?) remains open.

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