notes on OEIS A007925 https://oeis.org/A007925
a(n) = $n^{n+1}-(n+1)^n$ for n ≥ 0

by Mathew Englander

a(n) = A111454(n+4) – 1	https://oeis.org/A111454
a(n) = A055651(n, n+1)	https://oeis.org/A055651
a(n) = A220417(n+1, n), n ≥ 1	https://oeis.org/A220417
a(n) = A007778(n) – A000169(n+1)	https://oeis.org/A007778
	https://oeis.org/A000169

Compare:

https://oeis.org/A166326	Prime(n)^(prime(n)+1) - (prime(n)+1)^prime(n)
https://oeis.org/A099498	Semiprimes of the form A007925(n) = n^(n+1)-(n+1)^n
https://oeis.org/A141074	a(n) = n^(n+1)-(n+1)^n+1-(-1)^p(n+1)-(-1)^(n+1) where p(i) = i-th prime
https://oeis.org/A174379	a(n) is the largest prime factor of (n-1)^n - n^(n-1)
https://oeis.org/A123206	Primes of the form x^y - y^x, for x,y > 1
https://oeis.org/A045575	Nonnegative numbers of the form x^y - y^x, for x,y > 1
https://oeis.org/A082754	Triangle read by rows: T(n, k) = abs(n^k-k^n), 1<=k<=n

Theorems about divisibility of A007925

I. All a(n) are odd and
for n even, a(n) ≡ 3 mod 4
for n odd and n ≠ 1, a(n) ≡ 1 mod 4

II. Considering the values of n and a(n) mod 6:
for n ≡ 0, 1, 2, or 3, a(n) ≡ 5;
for n ≡ 4, a(n) ≡ 3;
for n ≡ 5, a(n) ≡ 1.

III. For n ≥ 0, a(n)+1 is a multiple of n^2.

IV. For n odd and n ≥ 3, a(n)–1 is a multiple of (n+1)^2;
for n even and n ≥ 0, a(n)+1 is a multiple of (n+1)^2.

Theorem I proof.
Considering the powers of m mod 4, we observe the following:
if m ≡ 0 then m^k ≡ 0 for all k ≥ 1;
if m ≡ 1 then m^k ≡ 1 for all k ≥ 0;
if m ≡ 2 then m^k ≡ 0 for all k ≥ 2;
if m ≡ 3 then m^k ≡ 1 for all even k and m^k ≡ 3 for all odd k, k ≥ 0.

The cases n=0 and n=1 are trivial: a(0) = a(1) = –1 which is odd and ≡ 3 mod 4. So now suppose n ≥ 2 and consider a(n) mod 4:
if n ≡ 0 then a(n) = n^(n+1) – (n+1)^n ≡ 0 – 1 ≡ 3;
if n ≡ 1 then a(n) = n^(n+1) – (n+1)^n ≡ 1 – 0 ≡ 1;
if n ≡ 2 then a(n) = n^(n+1) – (n+1)^n ≡ 0 – 1 ≡ 3 (because n is even);

if n ≡ 3 then a(n) = n^(n+1) – (n+1)^n ≡ 1 – 0 ≡ 1 (because n+1 is even).

Therefore all a(n) are odd and for n even, a(n) ≡ 3 mod 4, and for n odd and n ≠ 1, a(n) ≡ 1 mod 4. Q.E.D.

Theorem II proof.
Considering the powers of m mod 6, we observe the following:
if m ≡ 0 then m^k ≡ 0 for all k ≥ 1;
if m ≡ 1 then m^k ≡ 1;
if m ≡ 2 then m^k ≡ 4 for k even and k ≥ 2, m^k ≡ 2 for k odd;
if m ≡ 3 then m^k ≡ 3 for all k ≥ 1;
if m ≡ 4 then m^k ≡ 4 for all k ≥ 1;
if m ≡ 5 then m^k ≡ 1 for k even, m^k ≡ 5 for k odd.

For the cases n=0, n=1, and n=2, we have a(n) = –1 ≡ 5 mod 6. Now suppose n > 2 and consider n and a(n) mod 6:
if n ≡ 0 then a(n) = n^(n+1) – (n+1)^n ≡ 0 – 1 ≡ 5;
if n ≡ 1 then a(n) = n^(n+1) – (n+1)^n ≡ 1 – 2 ≡ 5 (because n is odd);
if n ≡ 2 then a(n) = n^(n+1) – (n+1)^n ≡ 2 – 3 ≡ 5 (because n+1 is odd);
if n ≡ 3 then a(n) = n^(n+1) – (n+1)^n ≡ 3 – 4 ≡ 5;
if n ≡ 4 then a(n) = n^(n+1) – (n+1)^n ≡ 4 – 1 ≡ 3 (because n is even);
if n ≡ 5 then a(n) = n^(n+1) – (n+1)^n ≡ 1 – 0 ≡ 1 (because n+1 is even).

Therefore, considering the values of n and a(n) mod 6:
    for n ≡ 0, 1, 2, or 3, a(n) ≡ 5;
    for n ≡ 4, a(n) ≡ 3;
    for n ≡ 5, a(n) ≡ 1.
Q.E.D.

Theorem III proof.
For n = 0, 1, or 2 we have a(n)+1 = 0, which is a multiple of n^2. Now suppose n > 2 and consider the binomial expansion of (n+1)^n:

$$n^n+\binom{n}{1}n^{n-1}+\binom{n}{2}n^{n-2}+\ldots+\binom{n}{n-2}n^2+\binom{n}{n-1}n+1$$

The penultimate term, $\binom{n}{n-1}n$, is equal to n^2. Every term to the left of that one is a multiple of n^2. It’s only the rightmost term, 1, that is not a multiple of n^2. Therefore we have (n+1)^n ≡ 1 mod n^2.

Because n > 2, we can say n^(n+1) ≡ 0 mod n^2.

Now a(n)+1 = n^(n+1) – (n+1)^n + 1 ≡ 0 – 1 + 1 ≡ 0 mod n^2.

Therefore for all n ≥ 0, a(n)+1 is a multiple of n^2. Q.E.D.

Theorem IV proof.
For n=0 and n=2, we have a(n)+1 = 0, which is a multiple of (n+1)^2.
The theorem does not apply to n=1. So now suppose n > 2. Let m = n+1.

Now consider $(m-1)^m \bmod m^2$. First look at the binomial expansion of (m – 1)^m:

$$m^m - \binom{m}{1} m^{m-1} + \binom{m}{2} m^{m-2} - \ldots \pm \binom{m}{m-2} m^2 \pm \binom{m}{m-1} m \pm 1$$

The rightmost term in this expansion is +1 if m is even, and –1 if m is odd. The penultimate term, $\pm\binom{m}{m-1} m$, is ±m^2. All the terms to the left of that one are multiples of m^2. So we have (m–1)^m ≡ 1 if m is even, –1 if m is odd, mod m^2.

Also, m^(m–1) ≡ 0 mod m^2. (We can say this because m > 3, since n > 2 and m=n+1.)

Therefore (m–1)^m – m^(m–1) ≡ +1 if m is even, –1 if m is odd, mod m^2.

And since m=n+1, we now have:

a(n) ≡ +1 if n is odd, –1 if n is even, mod (n+1)^2, for all n > 2.

Therefore:
For n odd and n ≥ 3, a(n)–1 is a multiple of (n+1)^2;
for n even and n ≥ 0, a(n)+1 is a multiple of (n+1)^2.
Q.E.D.

Combining theorems III and IV, we note that for even n, a(n) + 1 is a multiple of $n^2(n+1)^2 = n^4 + 2\mathrm{n}^3 + n^2$.

For example:
a(4) + 1 = 400, which is 16 * 25
a(6) + 1 = 162288, which is 36 * 49 * 92
a(8) + 1 = 91171008, which is 64 * 81 * 17587
a(10) + 1 = 74062575400, which is 100 * 121 * 6120874
a(12) + 1 = 83695120256592, which is 144 * 169 * 3439148597
(note that 3439148597 is prime)