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# TOWER OF HANOI WITH MORE PEGS

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2 to enter!

The traditional Tower of Hanoi puzzle consists of three pegs with 64 different sized rings stacked on one of them. The rings are arranged in decreasing order with the larger one on the bottom and the smallest one on the top of the stack. The problem is to move the rings one at a time from one peg to another, never putting a larger one on a smaller one, and eventually transferring all 64 rings from one peg to another peg.

The problem proposed for solution here is: Given $r$ pegs with $N$ rings arranged in decreasing order of size on one peg, what is the minimum number of moves required to arrange the $N$ rings in decreasing order of size on another peg while never having a larger ring on a smaller ring?

If the number of pegs is infinite, one can put $N - 1$ rings each on a single peg, then move the bottom ring and replace the $N - 1$ rings on it. This requires $2N - 1$ moves. Any time the number of pegs is greater than the number of rings, the situation is equivalent to having an infinite number of pegs and hence the answer is $2N - 1$.

For three pegs, there is a well-known result that the minimum number of moves required is $2^N - 1$ [1]. In proving this result by mathematical induction for $N + 1$ rings, one first moves $N$ rings to another peg ($2^N - 1$ moves), then moves the bottom ring to another peg (one move), and finally replaces the $N$ rings on top of this ($2^N - 1$ moves) to obtain the new total $2^{N+1} - 1$. If $f(N)$ is the number of moves for $N$ rings, the recursion relation resulting from this procedure is

$$f(N + 1) = 2f(N) + 1.$$

## Tower of Hanoi With Four Pegs

If the same procedure is followed for four pegs, there is the same recursion relation for $N \geq 3$. The first three values result from the formula for an infinite number of pegs as noted above. Hence we have Table 1.

Table 1.

| $N$ | $f(N)$ | $N$ | $f(N)$ |
|---|---|---|---|
| 1 | 1 | 6 | 47 |
| 2 | 3 | 7 | 95 |
| 3 | 5 | 8 | 191 |
| 4 | 11 | 9 | 383 |
| 5 | 23 | 10 | 767 |

not minimal

It is not difficult to show that these results do not provide the minimum number of moves. The alternative procedure is to make two piles. The first is constructed with four pegs available; the second has only three pegs that may be used since all the rings on the first peg are smaller than those that follow. To examine the various alternatives we take partitions of $N - 1$ into two parts. But since the number of moves for $k$ rings with four pegs is less than or equal to the number for three pegs, we need only examine partitions where the first part is greater than or equal to the second. Thus with four rings, the only alternative is the partition 2, 1. Since it takes three moves to build the tower of two rings with four pegs and one move for the tower of one ring with three pegs, the total number of moves is: $3 + 1 + 1 + 1 + 3 = 9$. If $n_4$ is the number of moves to build the first tower using four pegs and $n_3$ the number to build the second tower using three pegs, then the number of moves is

$$2(n_4 + n_3) + 1 .$$

Some additional cases are shown in Table 2.

Table 3 summarizes the results obtained up to 64 rings using this procedure. The same results (through $N = 11$) were obtained by Roth [2]. The first difference is entered for purposes of further analysis.

From the data we note that the difference is always a power of 2 and that the change in the power occurs at a quantity:

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

To arrive at a formula, let $n$ be chosen so that

$$\binom{n}{2} \leqq N < \binom{n+1}{2}$$

and $k$ be the quantity

$$k = N - \binom{n}{2}.$$

Table 2.

| | $n_3$ | $n_4$ | $f(N)$ |
|---|---|---|---|
| *Five Rings* | | | |
| 3,1 | 5 | 1 | 13* |
| 2,2 | 3 | 3 | 13* |
| *Six Rings* | | | |
| 4,1 | 9 | 1 | 21 |
| 3,2 | 5 | 3 | 17* |
| *Seven Rings* | | | |
| 5,1 | 13 | 1 | 29 |
| 4,2 | 9 | 3 | 25* |
| 3,3 | 5 | 7 | 25* |
| *Eight Rings* | | | |
| 6,1 | 17 | 1 | 37 |
| 5,2 | 13 | 3 | 33* |
| 4,3 | 9 | 7 | 33* |

* means a minimum.

The minimum number of moves according to the manner in which the table builds up is:

$$1 + 2\left[\binom{3}{2} - \binom{2}{2}\right] + 2^2\left[\binom{4}{2} - \binom{3}{2}\right] + 2^3\left[\binom{5}{2} - \binom{4}{2}\right] + \cdots$$

$$2^{n-2}\left[\binom{n}{2} - \binom{n-1}{2}\right] + 2^{n-1}k$$

$$= \sum_{k=2}^{n} 2^{k-2}\binom{k}{2} - \sum_{k=3}^{n-1} 2^{k-1}\binom{k-1}{2} + 2^{n-1}k.$$

Using the formula

$$\sum_{k=1}^{n}\binom{k}{2}2^k = 2^{n+1}\left[\binom{n+1}{2} - 2(n+1) + 4\right] - 4$$

The minimum number of moves becomes:

$$2^{n-1}\left[\binom{n+1}{2} - 2(n+1) + 4\right] - 1 - 2^{n-1}\left[\binom{n}{2} - 2n + 4\right] + 2 + 2^{n-1}k$$

$$= 2^{n-1}\,[n + k - 2] + 1.$$

Tower of Hanoi with 4 pegs.

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Table 3.

| N | f(N) | Δf(N) | N | f(N) | Δf(N) | N | f(N) | Δf(N) | N | f(N) | Δf(N) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 17 | 193 | 32 | 33 | 1409 | 128 | 49 | 6145 | 512 |
| 2 | 3 | 2 | 18 | 225 | 32 | 34 | 1537 | 128 | 50 | 6657 | 512 |
| 3 | 5 | 4 | 19 | 257 | 32 | 35 | 1665 | 128 | 51 | 7169 | 512 |
| 4 | 9 | 4 | 20 | 289 | 32 | 36 | 1793 | 256 | 52 | 7681 | 512 |
| 5 | 13 | 4 | 21 | 321 | 64 | 37 | 2049 | 256 | 53 | 8193 | 512 |
| 6 | 17 | 8 | 22 | 385 | 64 | 38 | 2305 | 256 | 54 | 8705 | 512 |
| 7 | 25 | 8 | 23 | 449 | 64 | 39 | 2561 | 256 | 55 | 9217 | 1024 |
| 8 | 33 | 8 | 24 | 513 | 64 | 40 | 2817 | 256 | 56 | 10241 | 1024 |
| 9 | 41 | 8 | 25 | 577 | 64 | 41 | 3073 | 256 | 57 | 11265 | 1024 |
| 10 | 49 | 16 | 26 | 641 | 64 | 42 | 3329 | 256 | 58 | 12289 | 1024 |
| 11 | 65 | 16 | 27 | 705 | 64 | 43 | 3585 | 256 | 59 | 13313 | 1024 |
| 12 | 81 | 16 | 28 | 769 | 128 | 44 | 3841 | 256 | 60 | 14337 | 1024 |
| 13 | 97 | 16 | 29 | 897 | 128 | 45 | 4097 | 512 | 61 | 15361 | 1024 |
| 14 | 113 | 16 | 30 | 1025 | 128 | 46 | 4609 | 512 | 62 | 16385 | 1024 |
| 15 | 129 | 32 | 31 | 1153 | 128 | 47 | 5121 | 512 | 63 | 17409 | 1024 |
| 16 | 161 | 32 | 32 | 1281 | 128 | 48 | 5633 | 512 | 64 | 18433 | 1024 |

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For example, if $N = 49$, $\binom{10}{2} \leqq 49 < \binom{11}{2}$ and $49 - 45 = 4 = k$. Hence the minimum number of moves by this formula would be

$$2^9\,[10 + 4 - 2] + 1 = 6145,$$

in agreement with the table value.

### Tower of Hanoi With Five Pegs

The procedure is similar to that employed with four pegs except that now three piles are used: the first is made using five pegs ($n_5$ moves), the second with four pegs ($n_4$ moves), and the third with three pegs ($n_3$ moves). The total number of moves is $2(n_5 + n_4 + n_3) + 1$. Partitions are taken of $N - 1$ with non-increasing parts. Thus for $N = 12$, the alternatives to be considered are: (7, 3, 1), (7, 2, 2), (6, 4, 1), (6, 3, 2), (5, 5, 1), (5, 4, 2), (5, 3, 3), (4, 4, 3). At this point a more streamlined method was developed using a strip which was placed next to the table values for five pegs. On this strip going from bottom to top was the minimum number of moves to form two towers of $n$ rings using four and three pegs. At a certain stage in the development the table for five pegs was:

| $N$ | $f(N)$ | $N$ | $f(N)$ |
|---|---|---|---|
| 1 | 1 | 7 | 19 |
| 2 | 3 | 8 | 23 |
| 3 | 5 | 9 | 27 |
| 4 | 7 | 10 | 31 |
| 5 | 11 | 11 | 39 |
| 6 | 15 | | |

For the combined four and three pegs, the minimum number of moves to set up two piles of $n$ rings in all is shown in Table 4.

Table 4.

| $n$ | Moves | $n$ | Moves |
|---|---|---|---|
| 16 | 96 | 8 | 20 |
| 15 | 80 | 7 | 16 |
| 14 | 64 | 6 | 12 |
| 13 | 56 | 5 | 8 |
| 12 | 48 | 4 | 6 |
| 11 | 40 | 3 | 4 |
| 10 | 32 | 2 | 2 |
| 9 | 24 | | |

These values can be obtained from the table for four pegs as follows. To find the moves for $n = 16$, go to 17 in the table for four pegs, subtract 1 and divide by 2: $(193 - 1)/2 = 96$.

Suppose we wish to find the minimum number of moves for 12 rings using five pegs. Place this scale against the values for the five pegs so that $N + n = 11$ (Table 5).

The minimum is 23, so that the number of moves for 12 rings with five pegs is 47. Now shift the scale down one and find the minimum for 13. And so on. Very quickly one can build up Table 6 for five pegs.

The differences again are powers of two while the points at which the difference changes are given by $\binom{n}{3}$. To establish a formula find $n$ such that

$$\binom{n}{3} \leqq N < \binom{n+1}{3} \text{ and } k = N - \binom{n}{3}$$

Then the minimum number of moves as given by these tabular values is:

$$\sum_{k=3}^{n} 2^{k-3}\binom{k}{3} - \sum_{k=4}^{n-1} 2^{k-2}\binom{k-1}{3} + 2^{n-2}k.$$

Using the relation

$$\sum_{k=1}^{n} \binom{k}{3} 2^{k} = 2^{n+1}\left[\binom{n+1}{3} - 2\binom{n+1}{3} + 4(n+1) - 8\right] + 8.$$

Table 5.

| $N$ | $f(N)$ | $n$ | Moves | $f(N)$ + Moves |
|---|---|---|---|---|
| 1 | 1 | 10 | 32 | 33 |
| 2 | 3 | 9 | 24 | 27 |
| 3 | 5 | 8 | 20 | 25 |
| 4 | 7 | 7 | 16 | 23 |
| 5 | 11 | 6 | 12 | 23 |
| 6 | 15 | 5 | 8 | 23 |
| 7 | 19 | 4 | 6 | 25 |
| 8 | 23 | 3 | 4 | 27 |
| 9 | 27 | 2 | 2 | 29 |

5 pegs



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Table 6.

| N | f(N) | Δf(N) | N | f(N) | Δf(N) | N | f(N) | Δf(N) |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 21 | 127 | 16 | 41 | 543 | 32 |
| 2 | 3 | 2 | 22 | 143 | 16 | 42 | 575 | 32 |
| 3 | 5 | 2 | 23 | 159 | 16 | 43 | 607 | 32 |
| 4 | 7 | 4 | 24 | 175 | 16 | 44 | 639 | 32 |
| 5 | 11 | 4 | 25 | 191 | 16 | 45 | 671 | 32 |
| 6 | 15 | 4 | 26 | 207 | 16 | 46 | 703 | 32 |
| 7 | 19 | 4 | 27 | 223 | 16 | 47 | 735 | 32 |
| 8 | 23 | 4 | 28 | 239 | 16 | 48 | 767 | 32 |
| 9 | 27 | 4 | 29 | 255 | 16 | 49 | 799 | 32 |
| 10 | 31 | 8 | 30 | 271 | 16 | 50 | 831 | 32 |
| 11 | 39 | 8 | 31 | 287 | 16 | 51 | 863 | 32 |
| 12 | 47 | 8 | 32 | 303 | 16 | 52 | 895 | 32 |
| 13 | 55 | 8 | 33 | 319 | 16 | 53 | 927 | 32 |
| 14 | 63 | 8 | 34 | 335 | 16 | 54 | 959 | 32 |
| 15 | 71 | 8 | 35 | 351 | 32 | 55 | 991 | 32 |
| 16 | 79 | 8 | 36 | 383 | 32 | 56 | 1023 | 64 |
| 17 | 87 | 8 | 37 | 415 | 32 | 57 | 1087 | 64 |
| 18 | 95 | 8 | 38 | 447 | 32 | 58 | 1151 | 64 |
| 19 | 103 | 8 | 39 | 479 | 32 | 59 | 1215 | 64 |
| 20 | 111 | 16 | 40 | 511 | 32 | 60 | 1279 | |

the minimum number of moves for five pegs becomes:

$$2^{n-2}\left[\binom{n+1}{3} - 2\binom{n+1}{2} + 4(n+1) - 8\right] + 1 - 2^{n-2}\left[\binom{n}{3} - 2\binom{n}{2} + 4n - 8\right]$$

$$- 2 + 2^{n-2}k = 2^{n-2}\left[\binom{n}{2} - 2n + 4 + k\right] - 1.$$

Thus, for $N = 53$,

$$35 = \binom{7}{3} < 53 < \binom{8}{3} = 56, \text{ and } k = 53 - 35 = 18.$$

The minimum number of moves for five pegs is thus

$$2^5\,[21 - 14 + 4 + 18] - 1 = 927.$$

If the pattern found for four and five pegs continues to hold for more pegs the conjecture is that the minimum number of moves for $r$ pegs and $N$ rings may be found as follows. Determine $n$ such that:

$$\binom{n}{r-2} \leqq N < \binom{n+1}{r-2} \quad \text{and} \quad k = N - \binom{n}{r-2}.$$

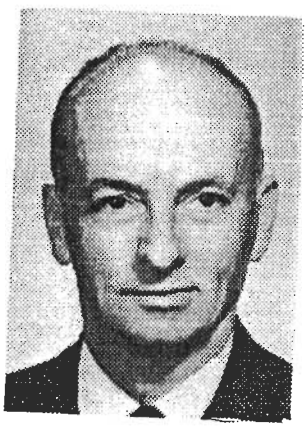
then the minimum number of moves should be:

$$2^{n-r+3}\left[\binom{n}{r-3} - 2\binom{n}{r-4} + 4\binom{n}{r-5} - 8\binom{n}{r-6} + \cdots + (-1)^{r-3}2^{r-3}\right] + (-1)^{r-2} + 2^{n-r+3}k.$$

## References


1. W. W. Rouse Ball and H. S. M. Coxeter, *Mathematical Recreations & Essays (12th Edition)*, University of Toronto Press, pp. 316-317, 1974.
2. T. Roth, "The Tower of Brahma Revisited," *J. Recreational Math.*, 7(2), pp. 116-119, Spring 1974.


### About the Author



Brother Alfred Brousseau was born in San Francisco, California in 1907. He obtained his A.B. and M.A. in mathematics from the University of California, Berkeley and his Ph.D. in physics from the same university in 1937. He is Professor of Mathematics at St. Mary's College in Moraga, California. His main interests are number theory, Fibonacci sequences, probability, and numerical analysis. He is an amateur botanist with a strong interest in California flora and the Fibonacci sequence in plants.

### ABC Puzzles – Puzzle A

"There is three errers in this sentence." Can you find them?

*Maxey Brooke, Sweeny, Texas*

NO

3 peg Hanoi

3, 7, 17, 41, 97, 225, 513, 1153, 2561, 5633, 12289

26625, 57345, 122881, 262145, 557057, 1179649,

4 peg: 4, 9, 21, 49, 113, 257, 577, 1281, 2817, 6145,

13313, 28673, 61441, 131073, 278529, 589825, 1245185.

5 peg: 5, 11, 25, 57, 129, 289, 641, 1409, 3073, 6657,

14337, 30721, 65537, 139265, 294913, 622593, 1310721

Neil, I have no idea where this came from

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Brousseau