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Numerical Solution of Laplace's Equation, Given Cauchy Conditions†

Usually Laplace's equation $\nabla^2\psi=0$ (or Poisson's equation $\nabla^2\psi=F$) must be solved with conditions given all around the boundary of the region in question. Yet in such specific engineering problems as the design of electron guns,[1,2] solutions are sought in an open region with the Cauchy boundary condition.** With Cauchy conditions, Laplace's equation is "unstable"[3] in that an exponential growth of errors occurs during numerical analysis by methods of finite differences. An expression that gives the order of magnitude of the propagated errors could therefore be of considerable value as a "rule of thumb" where these methods are used, particularly for digital computer programmers. This communication explains how this "rule of thumb" has been obtained.

The boundary C is, in general, not a straight line, nor can it be conformally mapped by analytic methods. Moreover, C may separate two different regions, respectively source-free (Laplace's equation) and source-present (Poisson's equation). When finite-difference methods are used, there are usually many "stars" which lie partially in each, no matter how fine the mesh is made. A typical five-point star,[1-7] $ABCDE$ in Fig. 1, has points in both the the Laplace region and the Poisson region (shaded area). To start a solution over the Laplace region, the potential ψ at mesh points of the first four diagonal lines is calculated by Taylor-series expansion of ψ near the boundary C. The potential ψ and its normal derivative ψ_n are given along C; partial derivatives of ψ along C are obtained by differentiating ψ, ψ_n along C and substituting in $\nabla^2\psi=0$ and its derivatives along and across C.[8] Once ψ is known on several "starting" lines, the five-point star formula,

$$\psi_4=4\psi_0-\psi_1-\psi_2-\psi_3\,,$$

can be used to calculate ψ at the rest of the mesh points to within the error of this difference approximation to the Laplacian operator. This technique of solving Laplace's equation by first determining several starting lines was originally suggested by Hyman.[9]

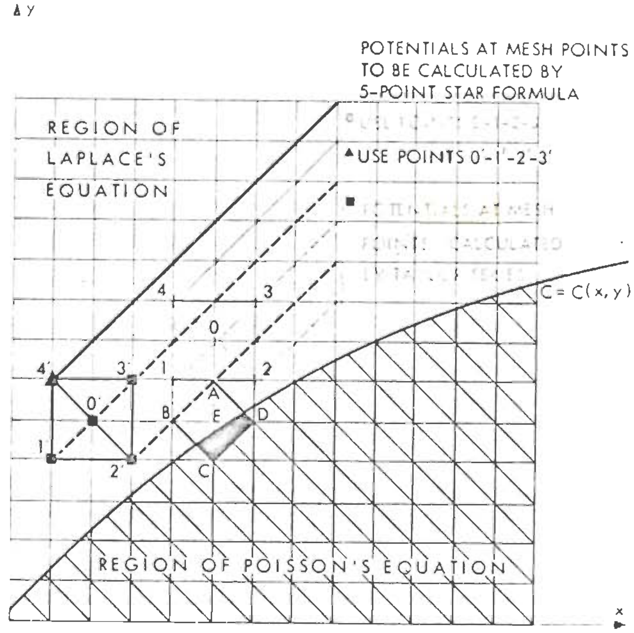

Figure 1 **General boundary and five-point star.**

Errors in the potentials at mesh points of the first four diagonal lines are inevitable because the Taylor-series expansions used to find them must be cut off after a finite number of terms, and the incremental distance for Taylor series (which is the mesh size) is small but never zero. The error at any point can be defined as

$$\varepsilon(x,y)=\psi_A(x,y)-\bar{\psi}(x,y)\,, \qquad (1)$$

where ψ_A is based on all the terms of the Taylor series and $\bar{\psi}$ on a finite number of terms. The implicit assumption of (1) is that when all the terms are used, the true potentials at mesh points in the vicinity of the boundary C can be obtained by Taylor-series expansions about potentials on C. Since even with multiple-precision arithmetic,[10] existing digital computers will have a limiting error larger than the sum of the terms neglected, this error may be used for $\varepsilon(x,y)$.

The build-up of errors in an open region occurs as follows: Let $\varepsilon_{1,i}$ and $\varepsilon_{2,i}$ be the known errors at mesh

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** … in proof: See F. John, "Notes on Improperly Posed Problems in Partial Differential Equations," Internal Report NN-117, Institute of Mathematical Sciences, New York University (August, 1958).

points of the first two lines; then accumulated errors at other mesh points are given as follows:

$$\varepsilon_{3,i} = -\varepsilon_{1,i} - \varepsilon_{2,i-1} + 4\varepsilon_{2,i} - \varepsilon_{2,i+1}\,, \tag{2}$$

where $i \geqslant 2$;

$$\varepsilon_{4,i} = -\varepsilon_{1,i-1} - 4\varepsilon_{1,i} + \varepsilon_{1,i+1} + \varepsilon_{2,i-2} - 8\varepsilon_{2,i-1} + 17\varepsilon_{2,i} - 8\varepsilon_{2,i+1} + \varepsilon_{2,i+2}\,, \tag{3}$$

where $i \geqslant 3$;

$$\varepsilon_{5,i} = -\varepsilon_{1,i-2} + 8\varepsilon_{1,i-1} - 17\varepsilon_{1,i} + 8\varepsilon_{1,i+1} - \varepsilon_{1,i+2} - \varepsilon_{2,i-3} + 12\varepsilon_{2,i-2} - 49\varepsilon_{2,i-1} + 80\varepsilon_{2,i} - 49\varepsilon_{2,i+1} + 12\varepsilon_{2,i+2} - \varepsilon_{2,i+3}\,, \tag{4}$$

where $i \geq 4$.

In general,

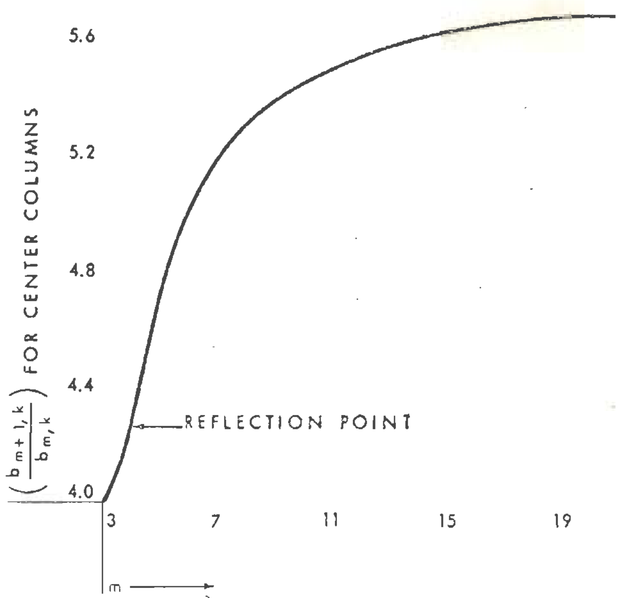
$$\varepsilon_{m,i} = \sum_{k=-(m-3)}^{k=+(m-3)} a_{m,k}\varepsilon_{1,i+k} + \sum_{k=-(m-2)}^{k=+(m-2)} b_{m,k}\varepsilon_{2,i+k}\,, \tag{5}$$

where $i \geq m-1$, k is the running index of summation, and $a_{m,k} = -b_{m-1,k}$ and $a_{m,k}$ and $b_{m,k}$ are integer coefficients.

Since errors on the second diagonal line of Fig. 1 are larger than those on the first diagonal line, the second summation dominates the magnitude of error in $\varepsilon_{m,i}$. The partial table of $b_{m,k}$ is shown below.

Since it is impossible to represent $b_{m,k}$ as a function of m and k, a graphical method was employed to investigate the gross rate of increase of the terms in the three central columns of the above table where the major coefficients occur. In Fig. 2 the ratio of each two successive center-column terms has been plotted against m. As m becomes larger than 10, the ratios asymptotically approach 5.7. The graphical study also showed that the magnitude of the terms in the two columns immediately adjacent to the center column rapidly begin to approach the magnitude

Figure 2 **Ratios of two successive terms of the center column.**



of the center-column terms. Thus the empirical expression

$$|\varepsilon_{m,i}| \leq |3\varepsilon_{2,i}(5.7)^{m-3}| \tag{6}$$

is obtained for the error bound.* This bound can be used by programmers in choosing a mesh size and number of figures carried. Equation (6) interrelates the mesh size and number of steps m before the accumulated error reaches this error bound.

*Kunz reports a similar exponential growth of error, considering only the center column.[11]

$m=2$						1					
$m=3$					−1	4	−1				
$m=4$				1	−8	17	−8	1			
$m=5$			−1	12	−49	80	−49	12	−1		
$m=6$		1	−16	97	−280	401	−280	97	−16	1	
$m=7$	−1	20	−161	672	−1569	2084	−1569	672	−161	20	−1

References

1. J. R. Pierce, *Theory and Design of Electron Beams,* Van Nostrand, New York, 1954, Chapter 10.
2. D. E. Radley, "The Theory of the Pierce-type Electron Gun." *Journal of Electronics and Control,* 4, No. 2, 125-148 (February, 1958).
3. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics,* Vol. 1, McGraw-Hill, N. Y., 1953, pp. 703-706.
4. F. S. Shaw, *An Introduction to Relaxation Methods,* Dover, New York, 1953, p. 39.
5. A. D. Booth, *Numerical Methods,* Academic Press, New York, 1955, p. 117.
6. J. B. Scarborough, *Numerical Mathematical Analysis,* Third Edition, The Johns Hopkins Press, 1955, p. 312.
7. W. E. Milne, *Numerical Solution of Differential Equations,* John Wiley, New York, 1955, p. 134.
8. A. G. Webster, *Partial Differential Equations of Mathematical Physics,* Dover, New York, 1955, Chapter 6.
9. M. A. Hyman, "Non-iterative Numerical Solution of Boundary-Value Problems," *Applied Science Research,* B2, 325-351 (1952).
10. D. D. McCracken, *Digital Computer Programming,* John Wiley, New York, 1957, Chapter 16.
11. K. S. Kunz, *Numerical Analysis,* McGraw-Hill, New York, 1957, p. 294.

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