Proof of conjecture in A6702

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Let a_0, a_1, a_2, … be the coefficients of a periodic continued fraction with period m, representing a square root, and let p_0, p_1/q_1, p_2/q_2, … be its convergents, so $p_n = K(a_0, …, a_n)$ and $q_n = K(a_1, …, a_n)$, with K the continuant function.

Task: prove that

$p_{n+2m} = C.p_{n+m} + (-1)^{m+1}.p_n$

$q_{n+2m} = C.q_{n+m} + (-1)^{m+1}.q_n$

$C = 2.p_{m-1}$

for all n>0

1) First write p_{n+2m} in terms of p_{n+m} and p_{n+m-1}:

$p_{n+2m} = a_{n+2m}.p_{n+2m-1} + p_{n+2m-2}$

$= a_{n+2m}(a_{n+2m-1}.p_{n+2m-2} + p_{n+2m-3}) + p_{n+2m-2} = (a_{n+2m}.a_{n+2m-1} + 1).p_{n+2m-2} + a_{n+2m}.p_{n+2m-3}$

…

$p_{n+2m} = K(a_{n+m+1}, …, a_{n+2m}).p_{n+m} + K(a_{n+m+2}, …, a_{n+2m}).p_{n+m-1}$

Or, since the a are periodic with period m:

$p_{n+2m} = K(a_{n+1}, …, a_{n+m}).p_{n+m} + K(a_{n+2}, …, a_{n+m}).p_{n+m-1}$

2) Then also write p_n in terms of p_{n+m} and p_{n+m-1}:

$p_n = p_{n+2} – a_{n+2}.p_{n+1}$

$= p_{n+2} – a_{n+2}(p_{n+3} - a_{n+3}.p_{n+2}) = (a_{n+2}.a_{n+3} + 1).p_{n+2} – a_{n+2}.p_{n+3}$

…

$= K(a_{n+2}, …, a_{n+m-1}).p_{n+m} – K(a_{n+2}, …, a_{n+m}).p_{n+m-1}$ if m even, or

$= K(a_{n+2}, …, a_{n+m}).p_{n+m-1} – K(a_{n+2}, …, a_{n+m-1}).p_{n+m}$ if m odd

3) Combine the results of 1 and 2:

$p_{n+2m} + p_n = [K(a_{n+1}, …, a_{n+m}) + K(a_{n+2}, …, a_{n+m-1})]. p_{n+m}$ if m even, or

$p_{n+2m} – p_n = [K(a_{n+1}, …, a_{n+m}) + K(a_{n+2}, …, a_{n+m-1})]. p_{n+m}$ if m odd

4) To show that the factor $[K(a_{n+1}, …, a_{n+m}) + K(a_{n+2}, …, a_{n+m-1})]$ is constant over all n>0, consider Euler's rule for the computation of continuants, by taking the sum of all possible products of the coefficients, in which any number of pairs of consecutive coefficients are deleted. Note that in $K(a_{n+1}, …, a_{n+m})$, there are no terms where the pair (a_{n+1}, a_{n+m}) is deleted. The "missing" terms, that would result from deleting (a_{n+1}, a_{n+m}) and any number of other pairs, are all in $K(a_{n+2}, …, a_{n+m-1})$.

Therefore $[K(a_{n+1}, …, a_{n+m}) + K(a_{n+2}, …, a_{n+m-1})]$ can be computed by writing all m coefficients $a_{n+1}, …, a_{n+m}$ in a circle, and then applying Euler's rule on this circle.

Since the coefficients are periodic with period m, writing $a_{n+1}, …, a_{n+m}$ in a circle is equivalent to writing $a_1, …, a_m$ in a circle, so the result does not depend on n:

$K(a_{n+1}, …, a_{n+m}) + K(a_{n+2}, …, a_{n+m-1})] = K(a_1, …, a_m) + K(a_2, …, a_{m-1}) = C$

5) From 3 and 4 we have that:
$p_{n+2m} = C.p_{n+m} - p_n$ if m even, or
$p_{n+2m} = C.p_{n+m} + p_n$ if m odd
Exactly the same proof holds for the equivalent statement about q.

6) Continuing from 4:

$C = K(a_1, \ldots, a_m) + K(a_2, \ldots, a_{m-1})$
$= a_m.K(a_1, \ldots, a_{m-1}) + K(a_1, \ldots, a_{m-2}) + K(a_2, \ldots, a_{m-1})$

Or, since $a_m = 2.a_0$ and the coefficients $a_1, \ldots a_{m-1}$ are a palindrome:

$C = 2. [a_0.K(a_1, \ldots, a_{m-1}) + K(a_2, .., a_{m-1})]$
$= 2. K(a_0, \ldots, a_{m-1})$
$= 2.p_{m-1}$

Which completes the proof of the conjecture.