EEN PAK MET EEN KORTE BROEK

PAPERS PRESENTED TO

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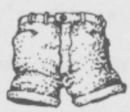

AMSTERDAM, MAY 18, 1977

SHE LOVES ME, SHE LOVES ME NOT

Relatives of two games of Lenstra

Richard K. Guy

H. W. Lenstra has suggested a method of generating games by turning coins. For example Sym is played on a row of coins, and a move is to turn any symmetrically arranged set of coins (e.g., the shaded ones in Fig. 1) provided that the rightmost of those turned

Figure 1. A Move in Sym.

goes from heads to tails (other coins may be turned in either sense). The purpose of this last condition is to ensure that the game finishes. In this game, and in all others in this paper, we adopt the Normal Play convention that a player unable to move loses, i.e. that the games are Last Player Winning. Sym, and indeed any Impartial Game (all of whose positions have the same set of options for each of the two players) is covered by the theory discovered independently by Sprague [6] and Grundy [4]. However the detailed strategy for playing most last player winning impartial games without ties (draws) is not known. Sym is mentioned in the chapter, Turn and Turn About, in [1] where it is stated that the nim-values (Sprague-Grundy functions) "display no recognizable pattern", those for a head-up coin in position n being exhibited as

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
nim-value	1	2	4	3	6	7	8	16	18	25	32	11	64	31	128	10	256	...

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As a start towards obtaining a complete analysis of this game, you are invited to investigate the game Sympler which is played like Sym except that the first coin on the left must always be included in the symmetrical pattern of coins turned in each move.

One can also play Lenstra's coin-turning games in two (or more!) dimensions. The game of Carpets is the cartesian product of two games of Sym, a typical move being to turn the coins (shaded in Fig. 2) at the intersections of a symmetrical set of rows with a symmetrical set of columns. To ensure that the game satisfies

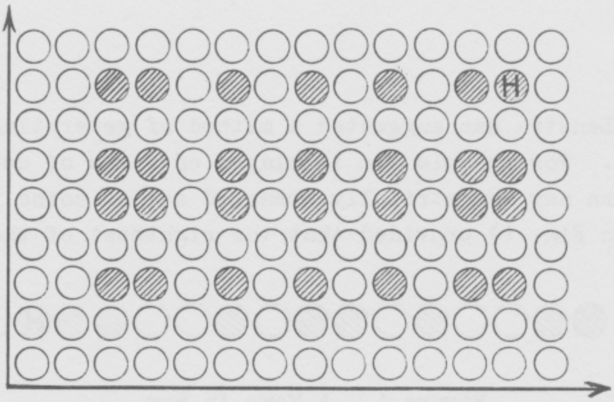

Figure 2. A Move in Carpets.

the Finishing Condition we also demand that the "most north-easterly" of the coins turned in any move goes from heads to tails. We are no better able to analyze Carpets than we were to analyze Sym. However we do know that the nim-value of a head-up coin in Carpets is the nim-product [2, p.52] of the nim-values corresponding to its two coordinates in Sym. Perhaps some light will be shed by the reader who is able to analyze Fitted Carpets, played like Carpets, but the carpet must fit snugly against the coordinate walls and include the "most south-westerly" coin at the origin.

For our next set of examples we turn to the "octal" games introduced by Guy and Smith [5]. These are played with heaps of beans, and a move (for either player) may be described by an octal code-name,

$$a_0 \cdot a_1 a_2 a_3 a_4 \ldots$$

where the code-digit a_r, $r > 0$, indicates the circumstances in which either player may remove r beans from a heap of n.

a_r	When you can take $r > 0$ beans from a heap
0	Never.
1	Only if the r beans constitute the whole heap; $r = n$.
2	Only if there are more than r beans in the heap; $r < n$.
3	Provided there are r beans in the heap; $r \leq n$. 3 = 2 + 1.
4	Only if the remaining $n - r$ (> 1) beans are left as 2 non-empty heaps.
5	= 4 + 1.
6	= 4 + 2.
7	= 4 + 2 + 1.

$\mathbf{a}_0 = \mathbf{4}$ or $\mathbf{0}$ according as it is permitted or not to split a heap into 2 non-zero heaps without taking any beans.

Guy and Smith [**5**] gave complete analyses of a number of such games, the best known, perhaps, being **·77**, Kayles, and **·137**, Dawson's Kayles. However, the vast majority of such games remains a complete mystery. Readers may like to investigate some of the following examples:

- **·3** Take a bean from a heap.
- **·5** Take a bean if it's isolated, or from a larger heap which must be left as 2 non-empty heaps.
- **·7** Take a bean from a heap, possibly splitting the remainder of the heap into 2 heaps. (There are also the generalizations to "into 3 heaps", "into 4 heaps", etc.)
- **4·0** Split a heap into 2 non-empty heaps.
- **4·2** Split a heap into 2 non-empty heaps, or take a bean from a larger heap.
- $\mathbf{\cdot\dot{3}\dot{0}}$ Take any odd number of beans.
- **4·01**, **4·04**, **4·05**, **4·21**, **4·24**, **4·25** and **·30X**, **·50X**, **·70X** where **X** is any one of the code-digits **1**, **2**, ... **7**.

The usual hope in investigating such games is that one is able to establish the periodicity of the nim-values.

For our next two games we turn to the chapter, Spots and Sprouts, in [**1**]. **Brussels Sprouts** was described in [**3**; Of Sprouts and Brussels Sprouts, games with a topological flavor, **217** No.1 (July 1967) 112-115; No.2 (Aug.) 109]. It was invented, along with Sprouts, by J.H. Conway and M.S. Paterson. The game starts with a number of 4-arm crosses. A move is to connect 2 arms (of the same cross or of different ones) by a continuous curve (which mustn't intersect any other curve or cross) and to make a new cross somewhere along its length, as in Figure 3. Since 2 arms are used in

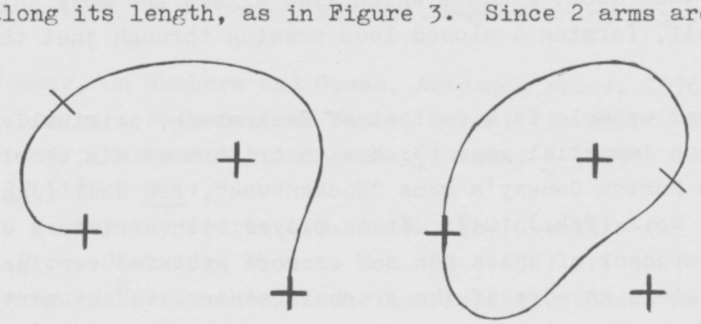

Figure 3. Two Possible First Moves in 3-cross Brussels Sprouts.

each move and 2 new ones created, it seems possible that the game can go on for ever. However if we start from n crosses and make m moves, we have $n + m$ crosses, $2m$ arcs joining them and $4n$ arms left, so there are at most $4m$ regions and Euler would be able to give a bound on m, and, in theory, the game can be analyzed. Mention of Euler reminds us that we might also play Brussels Sprouts on surfaces of higher genus, but the analysis of such a game is even more difficult as there seems to be no guarantee that the players will make use of the greater facilities available. A similar difficulty occurs on non-orientable surfaces, in addition to the physical problems which arise. Is it of advantage to use both sides of the paper?

Before considering the second game of this type we describe Lucasta, named in [1] for its inventor, Edouard Lucas. Start with a number of spots. A move is to draw a curve with 2 distinct spots as endpoints. These may not be the 2 endpoints of a single previously drawn curve, though they may be linked together by a chain of curves through intermediate spots. No 2 curves may cross, and no spot may be the endpoint of more than 2 curves, so the curves can only build up into chains or closed loops joining 3 or more spots. In [1] we give a strategy that enables you to win all the Lucasta positions you deserve to, when the number of chains is small. Indeed, we are able to give such a strategy for Misère Lucasta (Last Player Losing), a happy but complicated exception to the usual intractability of Misère Play (see [2], Chapter 12).

The game of Cabbages, or Bugs-Caterpillars-and-Cocoons, is like Lucasta but allows the move completing a closed loop passing through only 2 spots consisting of 2 curves with the same 2 endspots. It turns out that the analysis is already included in that for Lucasta. But what about *Jocasta* which also allows the move joining a spot to itself, forming a closed loop passing through just that one spot?

Our final example is a variant of Hackenbush, originally formulated as an impartial game [3; How to triumph at nim by playing safe, and John Horton Conway's game "Hackenbush", **226** No.1 (Jan. 1972) 104-107; No.2 (Feb.) 104]. It is played with a picture or graph, each component of which has one or more grounded vertices. A move is to delete an edge of the graph together with any part of the graph which is thus disconnected from the ground. For example

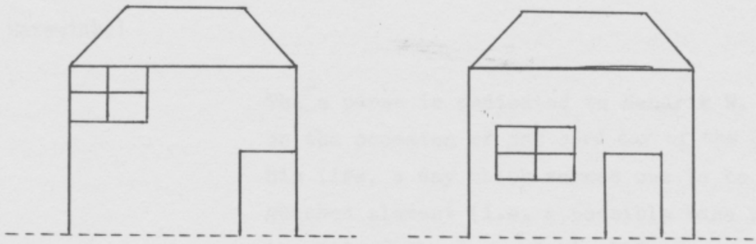

Figure 4. Childish and Adult Pictures of a House.

the adult picture in Fig. 4 is illegal; the window should be removed since it is not connected to the ground. Complicated but complete analyses of Hackenbush have been given by Conway [2, Chap.13] and Berlekamp [1]. It is perhaps too much to expect such an analysis for the Partizan version, Red-Blue Hackenbush, in which the edges are colored and Left may only delete blue edges and Right red ones, though disconnected edges of either color are removed. There may be more hope for an analysis of Childish Hackenbush, suggested by Jonathan Schaer, but even that game is not nearly as trivial as it might first appear. In Childish Hackenbush you may not delete edges which would disconnect other edges from the ground. We know that the values of Red-Blue Hackenbush positions are numbers, and can have arbitrarily large powers of 2 in their denominators. The values of Childish Hackenbush positions are also numbers. Richard Austin has found one with denominator 8, but it is not known if the denominators can be arbitrarily large. Can anyone throw light on Impartial Childish Hackenbush?

I am indebted to J.H. Conway for numerous conversations and for his suggestion that I was the right person to fill this much needed gap in game theory literature; there are many ways in which this paper would have been poorer had I not failed to heed his advice.

1. E.R. Berlekamp, J.H. Conway and R.K. Guy, Winning Ways, Freeman, 1977.

2. J.H. Conway, On Numbers and Games, Academic Press, 1976.

3. Martin Gardner, Mathematical Games, Scientific Amer., each issue.

4. P.M. Grundy, Mathematics and games, Eureka, 2 (1939) 6-8; reprinted ibid. 27 (1964) 9-11.

5. Richard K. Guy and Cedric A.B. Smith, The G-values for various games, Proc. Cambridge Philos. Soc. 52 (1956) 514-526; Zbl.74,345.

6. R.P. Sprague, Über mathematische Kampfspiele, Tôhoku Math. J., 41 (1935-36) 438-444; Zbl.13,290.