Scan Reshi 5388

ENUMERATING PARTITIONAL MATROIDS

538/

A. RECSKI

"He tried counting sheep which is sometimes a good way of getting to sleep..."

A. Milne: Winnie-the-Pooh

The concept of partitional matroids (the direct sum of uniform matroids) was first introduced by Berge [1]. Certain properties of partitional matroids were treated in [4], emphasizing also their connection with transversal matroids and their applicability in the analysis of electric networks. If a set \underline{S} has \underline{n} labelled elements and a partition of \underline{S} is considered then the number of different partitional matroids over this partition is at least \underline{n} -1 and at most $\underline{2n}$. The generating function for the number $\underline{X}_{\underline{n}}$ of partitional matroids of \underline{S} is determined and the result $\underline{n}^{-1}\log \underline{X}_{\underline{n}} = \log \underline{n} - \log\log \underline{n} - 1 + o(1)$ is obtained, which implies that the average number of partitional matroids over a partition is exp $(o(\underline{n}))$.

Let \underline{S} denote a finite set with \underline{n} labelled elements. $\mathcal{Y} = \underline{S_1, S_2, \ldots, S_p}$ is called a partition of \underline{S} if $\underline{U}_1 \, \underline{S}_1 = \underline{S}$ and if $\underline{S}_1 \, \underline{N} \, \underline{S}_j \neq 0$ if and only if $\underline{i} = \underline{j}$. $\mathcal{Y} = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_p)$ is an ordered set of integers such that $0 \leq \underline{a}_1 \leq |\underline{S}_1|$ ($\underline{i} = 1, 2, \ldots, p$). The system $[\mathcal{Y}, \mathcal{Y}]$ corresponds to a matroid on \underline{S} , namely a subset $\underline{X} \subseteq \underline{S}$ is independent if and only if $|\underline{X} \cap \underline{S}_i| \leq \underline{a}_i$ for $\underline{i} = 1, 2, \ldots, p$. The matroids of this type are called partitional matroids. In order to establish a one-one correspondence between the set of partitional matroids and the set of certain systems $[\mathcal{Y}, \mathcal{A}]$

the convention of [4] is applied again:

All of the subsets \underline{S}_i with $\underline{a}_i = 0$ or with $\underline{a}_i = |\underline{S}_i|$ are supposed to contain a single element only.

If \underline{A}_n is a sequence of numbers then let the symbol $\underline{A}_n\varepsilon\mathbb{A}$ abbreviate that

 $\underline{n}^{-1}\log \underline{A}_n = \log \underline{n} - \log\log \underline{n} - 1 + o(1).$

If \underline{G}_n denotes the number of partitions of the set \underline{S} with \underline{n} elements (the \underline{n}^{th} Bell-number) then $\underline{G}_n \varepsilon \Lambda$, see e.g. [2]. Our statement that $\underline{X}_n \varepsilon \Lambda$ too, will directly be implied by the following theorem, since $\sum_{n=0}^{\infty} \frac{G_n t^n}{n!} = \exp(e^t - 1)$ and $\exp(e^t \cdot p(t)) = o(\exp(e^{(1+\varepsilon)t}))$ for any polynomial p(t) and for any $\varepsilon > 0$.

THEOREM: $\sum_{i=0}^{\infty} \frac{X_n t^n}{n!} = \exp(e^t(t-1) + 2t + 1)$, if, by definition, $X_0 = 1$.

Proof: Let $\mathcal{F} = (\underline{S_1}, \underline{S_2}, \dots)$ be an arbitrary partition of \underline{S} and let $\underline{t_i}$ denote the number of the \underline{i} -element subsets in $\mathcal{F} = (\underline{s_1}, \underline{s_2}, \dots)$. The number of partitional matroids over $\underline{f} = (\underline{s_1}, \underline{s_2}, \dots)$ is obviously $\underline{f} = 1 + 1$ where $\underline{f} = 1 + 1$ if $\underline{f} > 1$ (cf. the above convention). The number of partitions of the labelled set, belonging to a certain system $\underline{T} = (\underline{t_1}, \underline{t_2}, \dots)$ is obviously

$$\frac{n!}{n} \quad \text{where } \Sigma i t_i = n. \text{ Thus, there are } i = 1 \quad t_i ! (i!)$$

partitional matroids over all the partitions belonging to \underline{T} .

Hence

$$\frac{X_nt^n}{n!} = \sum_{i=1}^n \frac{\left(\frac{x_it^i}{i!}\right)}{t_i!}$$

where the summation is over all systems of integers \underline{t}_i with n Σ i.t_i=n. Finally \underline{t}_i \underline{t}_i

which leads to the assertion since $\varphi_1(t) = e^{2t}$ and $\prod_{i=2}^{\infty} \varphi_i(t) = \exp(\sum_{i=2}^{\infty} \frac{t^i}{(i-1)!} - \sum_{i=2}^{\infty} \frac{t^i}{i!}) = \exp((t-1)e^t + 1).$

REMARKS: 1. Putting $\psi_1 = \psi_2 = \dots = 1$ the right hand side of (1) leads to the number \underline{G}_n of the partitions of \underline{S} (see Rényi [5]).

(see [4]) is considered, i.e. $\mathcal{M}_1 \subseteq \mathcal{M}_2$ if any independent subset of the matroid \mathcal{M}_1 is independent in \mathcal{M}_2 too. Let \underline{Y}_n denote the size of the longest antichain (i.e. the maximal number of pairwise incomparable elements) with respect to this partial ordering of the partitional matroids of \underline{S} . We prove that $\underline{Y}_n \in \Lambda$. The upper bound is trivial, since $\underline{Y}_n < \underline{X}_n$. The lower estimation $\underline{n}^{-2}\underline{G}_n < \underline{Y}_n$ will directly be implied by the pigeonhole-principle. Let $\underline{S}(\underline{n},\underline{k})$ denote the Stirling numbers of the second kind (the number of partitions of \underline{n} labelled elements into \underline{k} subsets). The relation $\underline{max} \ \underline{S}(\underline{n},\underline{k}) \in \Lambda$ is wellknown, see e.g. [3], and the $\underline{l} \le \underline{k} \le \underline{n}$

matroids of the same rank, proved in [4], leads to the assertion.

I wish to acknowledge some very helpful correspondence with G. Szekeres and V. V. Menon.

RESEARCH INSTITUTE FOR TELECOMMUNICATION 1026 BUDAPEST, GÁBOR Á. U. 65-67

References

/1/ C.Berge: Graphes et hypergraphes, Dunod, Paris, 1970.

/2/ F. Binet - G. Szekeres: On Borel fields over finite sets,

Ann. Math. Stat., Vol. 28. /1957/ pp. 494-498.

/3/ V. V. Menon: On the maximum of the Stirling numbers of

the second kind, J. Comb. Th. /A/ Vol. 15, No. 1. pp. 11-24.

/4/ A. Recski: On partitional matroids with applications, to

appear in the Proceedings of the Keszthely Colloquium, 1973.

/5/ A. Rényi: Új módszerek és eredmények a kombinatorikus

analizisben, MTA. III. oszt. közl. Vol. 16/1966/ pp. 77-105.