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UNIQUENESS IN FINITE MEASUREMENT

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Abstract. This article surveys recent investigations of real sequences $(d_1, \ldots, d_n)$ which arise in the theory of measurement from considerations of uniqueness for numerical representations of qualitative relations on finite sets. The sequences we discuss arise from measurement problems which include measurement of subjective probability, extensive measurement, difference measurement, and additive conjoint measurement. The measurement problems lead to sequences with fascinating combinatorial and number–theoretic properties.

The unifying mathematical framework under which we analyze uniqueness of measurement in these diverse areas involves the analysis of the sequences $(d_1, \ldots, d_n)$ as the solutions of finite systems of linear equations. Different applications are translated into different restrictions on the types of linear equations that are admissible for each area.

Two primary concerns of measurement theory are involved in the work being surveyed: (1) axioms for the qualitative relation that are necessary and sufficient, or at least sufficient, for unique representability; (2) the structure of sets of unique solutions. The latter concern leads to combinatorial and number–theoretic problems involving characterizations of unique solutions, counts of numbers of unique solutions, and extreme–value questions. Definitive results and presently open problems are described for the areas covered by the basic theory.

1. Introduction

In this paper we consider a series of problems in combinatorics and number theory which arise in the representational theory of measurement (Fishburn [1970, 1988], Krantz, et al. [1971], Narens [1985], Pfanzagl [1968], Roberts [1979]).

In measurement theory, we are frequently interested in mappings u which assign a real number to each element of a given qualitative structure $\mathcal{A}$ in such a way that certain features of the qualitative structure are preserved. We pick certain properties that we would like such a mapping to have and call these a *model*. Sometimes the model is called a *representation* and sometimes it is the mapping which is called a *representation*. The theory is concerned with three kinds of questions: *The Modeling Question*: What properties define an appropriate model?; *The Existence Question*: Given a model, what properties must the qualitative structure have in order for there to be a mapping which satisfies it?; *The Uniqueness Question*: What are the relationships among all possible mappings that satisfy the model? In this paper we shall be primarily interested in the uniqueness question.

The theory of measurement has undergone immense development during the past 20 years, as evidenced by the books cited above. The majority of this work deals with the three questions we have posed for infinite qualitative relational structures, which under assumed continuity conditions tend to have "nice" answers to the existence and uniqueness questions for a variety of models. There is a growing literature on the existence question for finite structures, thanks in part to the

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Fishburn [1970], Theorem 4.2) then shows that there is a d solution for our sets if and only if B2 holds. Finally, B1 is used to conclude that $d_j > 0$ for all j.

Given the existence of a positive u solution for (9), equivalently a positive d solution for (17), Theorem 1 and the restrictions of $\mathfrak{B}_0$ say that d is unique if and only if there are $n-1$ linearly independent equations obtained from $\sim$ of the form

$$d_i = d_j \, (i \neq j) \quad \text{and} \quad d_i = d_j + d_k \, (i \neq j \neq k \neq i).$$

The existence of such a set can be proposed as a third axiom to complete our necessary and sufficient conditions for the existence of a mapping u that satisfies (9) and is unique up to proportionality transformations.

However, a simpler uniqueness condition can be stated once we understand the nature of the set of all unique d solutions. Assume $n \geq 2$ and for convenience fix $\min d_j$ at 1. Then it is easily verified that d is a unique solution for (17) if and only if there is a permutation $x_1, \ldots, x_n$ of $d_1, \ldots, d_n$ such that

$$x_1 \leq x_2 \leq \cdots \leq x_n,$$
$$x_1 = \cdots = x_t = 1 \quad \text{for some} \quad 2 \leq t \leq n,$$

and, for every i such that $t < i \leq n$,

$$x_i = x_j + x_k \quad \text{for some distinct} \quad j \quad \text{and} \quad k \quad \text{less than } i.$$

This leads to the uniqueness axiom

B3. *For all $i, i' \in \{1, \ldots, n\}$, if $\{i\} > \{i'\}$ then $\{i\} \sim \{j,k\}$ for some distinct $j, k \in \{1, \ldots, n\} \backslash \{i\}$,*

and to the following representation–uniqueness theorem.

THEOREM 2. *The relation $>$ on $\mathfrak{B}_0$ satisfies B1, B2 and B3 if and only if there is a unique (up to multiplication by a positive constant) positive d that satisfies (17) for all $\{A, B\} \in \mathfrak{B}_0$.*

Let E_n be the set of all n-term nondecreasing sequences $x_1, x_2, \ldots, x_n$ of positive integers that satisfy

$$x_1 = x_2 = 1,$$
$$x_i = x_j + x_k \quad \text{for some} \quad j \neq k \quad \text{whenever} \quad x_i > 1.$$

We refer to sequences in $E_2 \cup E_3 \cup \ldots$ as *elementary sequences*. As just noted, the set E_n of n-term elementary sequences represents the set of unique solutions for the $\mathfrak{B}_0$ model for n in the sense that, up to proportionality, d is a unique solution if and only if its components form a permutation of a sequence in E_n. Similar characterizations by sequences of positive integers in smallest–integer format will also be used in ensuing sections for solution sets encountered there.

4.2 Numbers of Solutions. The E_n characterization gives us access to the **counting problem** since it provides a convenient base for counting the number of different unique solutions. Table 1 gives both $|E_n|$ for $n \geq 2$ and the number of different solutions obtained from permutations of the components of each member of E_n. The table shows that even in a very restricted context there can be a large

TABLE 1

Counts for restricted extensive case (E_n)
and all sub–Fibonacci sequences (F_n)

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n	$\lvert F_n \rvert$	$\lvert E_n \rvert$	$\lvert E_n$ with permutations $\rvert$
2	1	1	1
3	2	2	4
4	4	4	23
5	10	10	256
6	31	31	4647
7	127	120	128,262
8	711	578	5,128,503

number of unique realizations of the representation for modest values of n. We do not have a workable recurrence for $|E_n|$ to extend the table's values to larger n, nor do we have an order–of–magnitude approximation of $|E_n|$ for large n, but such things could be considered in this and other contexts.

4.3 Extreme Values. An example of an **extremization problem** for elementary sequences is to determine the largest value of x_n for all sequences in E_n. The answer is obvious:

$$\max\{x_n : (x_1, \ldots, x_n) \in E_n\} = f_n,$$

where f_n is the n^{th} Fibonacci number, i.e., the n^{th} term in the Fibonacci sequence $f_1, f_2, f_3, \ldots$ defined by $f_1 = f_2 = 1$ and $f_j = f_{j-1} + f_{j-2}$ for all $j \geq 3$. There is a unique member of E_n that yields this maximum, namely $(f_1, f_2, \ldots, f_n)$, and this member of E_n also clearly maximizes the sum of the x_i. By the well-known Fibonacci identity

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1, \quad n \geq 1,$$

the maximum sum is $f_{n+2} - 1$.

We refer to a nondecreasing integer sequence $x_1, x_2, \ldots, x_n$ as a *sub–Fibonacci sequence* if $x_1 = x_2 = 1$ and $x_j \leq x_{j-1} + x_{j-2}$ for $j = 3, \ldots, n$ and let F_n denote the set of all n–term sub–Fibonacci sequences. It is easy to see that $E_n \subseteq F_n$. Table 1 notes that $n = 7$ is the smallest n at which $|E_n| < |F_n|$. The smallest sub–Fibonacci regular extensions are not Van Lier extensions. The y values for the latter are 27, 37, 50 and 53, but only 50 and 53 are values defined as in Theorem 10.

Similar to the definition of e_n^* in Section 4, let

$$v_n^* = \max\{|\{y : (x_1, \ldots, x_n, y) \in V_{n+1}\}| : (x_1, \ldots, x_n) \in V_n\},$$

the maximum number of one–term Van Lier extensions of a Van Lier sequence in V_n. It is easily checked that $v_n^* = f_{n+1}$ for the first few n. We know more.

THEOREM 11 ([FRM]). *Every one–term regular extension of $f_1, f_2, \ldots, f_{n-1}, x_n$ with $f_{n-1} \le x_n \le f_n$ is in V_{n+1}. Therefore*

$$v_n^* \ge f_1 + \cdots + f_{n-1} + 1 = f_{n+1}.$$

Investigation of v_n^* for small n leads to the conjectures that the bound in Theorem 11 is exact, and that the number of one–term Van Lier extensions of $(x_1, \ldots, x_n) \in V_n$ is precisely f_{n+1} if and only if $x_i = f_i$ for all $i < n$ and $f_{n-1} \le x_n \le f_n$. However, it turns out that the exactness of the bound holds only for $n \le 7$. At $n = 8$ the Van Lier sequence 1, 1, 2, 4, 6, 10, 13, 16 has $35 = f_9 + 1$ one–term Van Lier extensions. The latter conjecture holds for $n \le 6$, but fails at $n = 7$ since the Van Lier sequence 1, 1, 2, 4, 6, 10, 13 has $21 = f_8$ one–term Van Lier extensions but violates the properties indicated.

5.5 Unique Probability Sequences. Conjecture 1 is suggested by the next theorem.

THEOREM 12 ([FO]). *$|R_n|/|P_n| \to 0$ as $n \to \infty$.*

This shows that there are many more nondecreasing integer sequences in the solution set P_n for UP3 than in the regular set R_n for large n. The proof of Theorem 12 follows from the observation that 1234 is in $P_4 \setminus R_4$ (its three equations are $p_3 = p_1 + p_2$, $p_4 = p_1 + p_3$ and $p_1 + p_4 = p_2 + p_3$) and then that, as m gets large, the number of regular m-term extensions of 1234 overwhelms the number of regular m-term extensions for each of the six sequences in R_4.

We refer to sequences $P_n \setminus R_n$ as *irregular sequences*. There are two 4–term irregular sequences, 1223 and 1234, and 75 irregular sequences for $n = 5$. Table 2 summarizes counts for small n of the sequence types considered thus far. The blank spaces in the table have not been determined at present.

A sample of irregular sequences in $P_5 \setminus R_5$ includes

$$(1,1,3,3,5), \quad (2,2,2,3,3), \quad (4,5,6,7,8), \quad (1,3,6,8,10).$$

We invite readers to specify four linearly independent equations (in p_i or d_i or x_i) that correspond to each solution. The middle two sequences show that x_1 must exceed 1 in the smallest–integer format for some cases.

TABLE 2

Counts for restricted extensive (E_n), sub–Fibonacci (F_n),
Van Lier (V_n), Regular (R_n) and Irregular ($P_n \setminus R_n$)
Sequences with n terms

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n	$\lvert E_n\rvert$	$\lvert F_n\rvert$	$\lvert V_n\rvert$	$\lvert R_n\rvert$	$\lvert P_n \setminus R_n\rvert$	$\lvert P_n\rvert$
2	1	1	1	1	0	1
3	2	2	2	2	0	2
4	4	4	6	6	2	8
5	10	10	26	27	75	102
6	31	31	164	192		
7	120	127	1529	2280		
8	578	711	21,439	47,097		

The last of the preceding solutions, (1, 3, 6, 8, 10), shows that the smallest $p_i > 0$ in an irregular unique solution, which is $p_1 = x_1/\sum x_i = 1/28$ for this case, may be substantially smaller than the smallest $p_i > 0$ for R_n, which is $p_1 = 1/2^{n-1}$ for the regular sequence $1, 1, 2, 4, \ldots, 2^{n-2}$. More specifically, we have

THEOREM 13 ([FO]). *$\min\{x_1/\sum x_i : (x_1, \ldots, x_n) \in P_n\}/2^{n-1} \to 0$.*

The minimum value of $x_1/\sum x_i$ for P_5 is 1/28, for sequence (1,3,6,8,10). The smallest $x_1/\sum x_i$ for P_6 and P_7 presently known are 1/64 and 1/192, respectively. These arise from $(1, 4, 7, 10, 20, 22) \in P_6$ and $(1, 5, 14, 18, 36, 44, 74) \in P_7$.

Another extremization problem for irregular sequences examined in [FO] is to determine the largest ratio x_2/x_1 for sequences in P_n. Since $x_2/x_1 = 1$ for all regular sequences, it might be supposed that we can do somewhat better than this with irregulars.

THEOREM 14 ([FO]). *$\max_{P_n}(x_2/x_1) \ge 2^{n-4} + 2^{n-6} + 2^{(n-4)/2}$ for $n \ge 5$.*

Sequence $(1,4,5,6,8) \in P_5$ verifies the theorem at $n = 5$.

A third extremization problem, which is considered in [FMR], is to determine the maximum value of x_n/x_1 for sequences in P_n. We know from [FO] that $\max x_4/x_1 =$