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Struggling with the 3x + 1 problem

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A famous unsolved problem in mathematics is the $3x+1$ conjecture. An excellent discussion of its origin and history is found in [2]. It is deceptively simple to state. Many are tempted to try it because of its enticing appearance. One aim of this note is to warn the uninitiated that a person can grow old and infirm by trying to solve the problem. There are probably fine mathematicians who have frittered away what might have been productive periods of their lives working on this devilish puzzle. The other aim is to say something at least mildly interesting about a question that I can't answer. These are the writings of a person who has been frustrated, but not defeated, by the $3x+1$ conjecture.

Here is a statement of the problem:

Define a function f on the positive integers by

$$f(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ \dfrac{x}{2} & \text{if } x \text{ is even.} \end{cases}$$

The $3x+1$ conjecture states that for $x \geqslant 2$, $f^n(x) = 1$ for some n; i.e. applying f enough times will produce the answer 1. For example, the number 3 iterates to 1 after seven applications of f:

3, 10, 5, 16, 8, 4, 2, 1.

Certainly, here is a problem that a mathematics student at almost any level can work on. Generally, it turns out that a sixth former can contribute as much toward the solution as a research mathematician. This is distressing to the mathematician. He needs a way to save face.

Paul Halmos is fond of quoting a dictum of Polya: "If there is a problem you can't solve, then there is an easier problem that you can't solve. Find it." After applying this advice as often as necessary, one arrives at a solvable problem. Starting with the $3x+1$ problem, here is a problem I can solve—the $x+1$ problem:

Define a function g on the positive integers by

$$g(x) = \begin{cases} x+1 & \text{if } x \text{ is odd} \\ \dfrac{x}{2} & \text{if } x \text{ is even.} \end{cases}$$

It is easy to prove that for $x \geqslant 2$, $g^n(x) = 1$ for some n.

Perhaps I made the problem too easy and lost much of the spirit of the original question. There are, however, some noteworthy consequences of the definition. Notice that the largest number that iterates to 1 in k steps is 2^k. How many other numbers converge in k steps? For example, which numbers converge in 5 steps? A quick check shows that 32, 15, 14, 12, and 5 are the only ones which iterate to 1 after 5 applications of g. A question with an interesting answer is the following:

Let n be a positive integer. Define $Kg(n)$ to be the number of integers x so that $g^n(x) = 1$. Is there a formula for $Kg(n)$?

Notice that:

$Kg(1) = 1$, (2 is the only integer converging in 1 step)
$Kg(2) = 1$, (4 is the only one)
$Kg(3) = 2$, (8 and 3 converge in 3 steps)
$Kg(4) = 3$, (16, 7, 6)
$Kg(5) = 5$.

It doesn't take long to suspect and it is not hard to prove (see [1]) that $Kg(n)$ is the nth Fibonacci number!

Having made this discovery, one naturally asks what sequences are produced if there are modifications of the iterative function. Defining g on the positive integers which are not multiples of 3 by

$$g(x) = \begin{cases} x+3 & \text{if } x \text{ is odd} \\ \dfrac{x}{2} & \text{if } x \text{ is even} \end{cases}$$

again produces the Fibonacci numbers.

To produce the Lucas numbers, (produced like Fibonacci numbers, but starting 1, 3), define m on the positive integers which are not multiples of 5 by

$$m(x) = \begin{cases} x+5 & \text{if } x \text{ is odd} \\ \dfrac{x}{2} & \text{if } x \text{ is even.} \end{cases}$$

Let $Km(n)$ be the number of integers x so that $m^n(x) = 1$. Then $Km(1) = Km(2) = Km(3) = Km(4) = 1$, $Km(5) = 2$, $Km(6) = 3$, $Km(7) = 4$, $Km(8) = 7$, $Km(9) = 11, \ldots$. Notice that there was a slight problem in getting the Lucas numbers started, but after $Km(6)$, everything is fine.

In general, if a is an odd prime and one defines a function h on the positive integers which are not multiples of a by

$$h(x) = \begin{cases} x+a & \text{if } x \text{ is odd} \\ \dfrac{x}{2} & \text{if } x \text{ is even,} \end{cases}$$

then $Kh(n)$ leads to a Fibonacci-type sequence after a finite number of terms.

I am now ready to offer a way of neutralising the $3x+1$ problem. If the $x+1$ problem produces the Fibonacci sequence, then the $3x+1$ problem is merely a way of generating a generalised Fibonacci sequence! For n a positive integer let $Kf(n)$ be the number of integers x so that $f^n(x) = 1$. The first few values of $Kf(n)$ give the following sequence: 18 58, 72, 91, 113, 143

$$1, 1, 1, 1, 1, 2, 2, 4, 4, 6, 6, 8, 10, 14, \cancel{28}, 24, 29, 36, 44, \ldots$$

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This reclassification of the $3x+1$ problem would, perhaps, be more striking if one could produce a formula for obtaining the nth term of the sequence: I can't!

Actually, if one is interested in defining generalised Fibonacci sequences, there are some drawbacks to the way outlined above. Following an $x+1$ problem and a $3x+1$ problem would come a $5x+1$ problem. However, a $5x+1$ function leads quickly to cycles For example, starting with 13 one obtains the following:

$$13, 66, 33, 166, 83, 416, 204, 52, 26, 13, \ldots.$$

Let me conjecture a remedy for this defect. The $3x+1$ function is not properly defined. It should be redefined as follows:

$$f(x) = \begin{cases} \dfrac{x}{3} & \text{if } x \text{ is divisible by 3} \\ \dfrac{x}{2} & \text{if } x \text{ is divisible by 2 but not by 3} \\ 3x+1 & \text{otherwise.} \end{cases}$$

Based on intuition and a few examples, I believe the cycle problem mentioned above will be eliminated. Now I can't prove that iterates of my new function will not produce cycles. (If I could, I could probably prove the original $3x + 1$ conjecture.) In fact, I believe that the average candidate for the asylum will find that this modified function leads to a "$3x + 1$ conjecture" that is no less challenging than the original one. But, let me extend it.

In general, for k an odd prime, let $k_1, \ldots, k_n$ be a listing of all primes less than or equal to k, with $k_1 > k_2 > \ldots > k_n$. Define f_k on the positive integers by

$$f_k(x) = \begin{cases} \dfrac{x}{k_1} & \text{if } x \text{ is divisible by } k_1 \\ \dfrac{x}{k_2} & \text{if } x \text{ is divisible by } k_2 \text{ but not by } k_1 \\ \vdots & \\ \dfrac{x}{k_n} & \text{if } x \text{ is divisible by } k_n \text{ but not by } k_1 \text{ or} \ldots k_{n-1} \\ kx + 1 & \text{otherwise.} \end{cases}$$

Define $F(x)$ on the positive integers as follows:

$$F(x) = \lim_{k \to \infty} f_k(x).$$

Is it true that for every $x \geqslant 2$, $F^n(x) = 1$ for some n? Clearly yes. The fundamental theorem of arithmetic says that n would be the number of prime factors (with repetitions) of x.

Thus it is possible to solve an extended $3x + 1$ problem even though the original still eludes us. Perhaps one of your fifth or sixth formers will finish the job.

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"Even Gary Lineker's failure to hit the target can be seen as good news, for his success rate is such that one representative game without a goal increases the chances for his scoring in the next." From the *Glasgow Herald* of 10 August 1987, spotted by A. C. M. MacNeill.