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THE DERIVATIVES OF X^X

Richard R. Patterson and Gaurav Suri, Department of Mathematical Sciences, Indiana University - Purdue University at Indianapolis, Box 647, Indianapolis, Indiana, 46223.

While trying to find the Taylor expansion for x^x about 1, the second author (then a freshman) observed that the n'th derivative of x^x evaluated at $x = 1$ always seems to be divisible by n. Thus we have the conjecture

$$n \mid (x^x)^{(n)}(1) .$$

The conjecture has been verified for $n \leq 18$ using the symbolic computer algebra program REDUCE by Anthony C. Hearn of the University of Utah. Let $y_n = (x^x)^{(n)}(1)$.

n	y_n	
1	1	1
2	2	1
3	3	1
4	8	2
5	10	2
6	54	9
7	-42	-6
8	944	118
9	-5112	
10	47160	
11	-419760	

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12	4297512
13	-47607144
14	575023344
15	-7500202920
16	105180931200
17	-1578296510400
18	25238664189504

We can develop a recursion formula for y_n using Leibnitz's formula for the n'th derivative of a product:

$$(uv)^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \cdots + \binom{n}{r}u^{(n-r)}v^{(r)} + \cdots + uv^{(n)}.$$

The first derivative of x^x is $x^x(1 + \ln x)$. Let $u = x^x$ and $v = 1 + \ln x$. Then

$$(x^x)^{(n+1)} = (x^x)^{(n)}(1 + \ln x) + \binom{n}{1}(x^x)^{(n-1)}\frac{1}{x} - \binom{n}{2}(x^x)^{(n-2)}\frac{1}{x^2} + \cdots$$

$$+ (-1)^{r-1}\binom{n}{r}(x^x)^{(n-r)}\frac{(r-1)!}{x^r} + \cdots + (-1)^{n-1}x^x\frac{(n-1)!}{x^n}.$$

Can we better with $\frac{d^n}{dx^n}(x \ln x)$

Letting $x = 1$ (and also $y_o = 1$) we obtain the recursion formula

$$y_{n+1} = y_n + \binom{n}{1}y_{n-1} - \binom{n}{2}y_{n-2} + \binom{n}{3}2!y_{n-3} + \cdots + (-1)^{r-1}\binom{n}{r}(r-1)!y_{n-r}$$

$$+ \cdots + (-1)^{n-1}(n-1)!y_o .$$