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A baker's dozen ... Preprint

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(good overview)

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A BAKER'S DOZEN OF CONJECTURES CONCERNING PLANE PARTITIONS

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see the next to last page for some sequences

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(preprint)

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Many remarkable conjectures have been made recently concerning the explicit enumeration of certain classes of tableaux. Most of these are due to or arise from the work of W. Mills, D. Robbins, and H. Rumsey. Here we will survey the most prominent of these conjectures (omitting some rather technical refinements). We will for the most part not discuss the background of these conjectures and their connections with symmetric functions and representation theory. We will also for the most part ignore a host of known results which are very similar to many of the conjectures and which make the conjectures considerably more tantalizing. The reader should consult the references cited below for further information.

We begin with the necessary definitions. A plane partition π is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers π_{ij} with finite sum $|\pi| = \Sigma\, \pi_{ij}$, which is weakly decreasing in rows and columns [10]. The nonzero π_{ij} are called the parts of π, and normally when writing examples only the parts are displayed. Such terminology as "number of rows of π" refers only to the parts of π. Thus, for example,

```
443211
43311
321
22
1
```

is a plane partition π with $|\pi| = 38$, and with 17 parts, 5 rows, and 6 columns. We now list some special classes of plane partitions.

column-strict: the parts strictly decrease in each column.

row-strict: the parts strictly decrease in each row.

symmetric: $\pi_{ij} = \pi_{ji}$ for all i,j.

cyclically symmetric: the i-th row of π, regarded as an ordinary partition, is conjugate (in the sense of [4, p. 21]) to the i-th column, for all i.

totally symmetric: symmetric and cyclically symmetric.

(r,s,t)-self-complementary: π has $\leq r$ rows, $\leq s$ columns, largest part $\leq t$, and $\pi_{ij} + \pi_{r-i+1,s-j+1} = t$ for all $1 \leq i \leq r$, $1 \leq j \leq s$.

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Example. Consider the three plane partitions

4431	4432	44321
3321	4331	4222
321	332	321
2	21	

The first is cyclically but not totally symmetric, while the second is totally symmetric. Moreover, the third is (3,5,4) - self-complementary.

A Gelfand pattern (see [3]) is a triangular array

$$\begin{matrix} a_{11} & a_{12} \cdots & a_{1n} \\ & a_{22} \; \cdots \; a_{2n} & \\ & \vdots & \\ & a_{nn} & \end{matrix}$$

of nonnegative integers a_{ij} which weakly increase in rows and such that $a_{i-1,j-1} \le a_{ij} \le a_{i-1,j}$ for all $2 \le i \le j \le n$. A Gelfand pattern is strict if the rows stricly increase. A strict Gelfand pattern with first row $1,2,\ldots,n$ is called a monotone triangle of length n. An $n \times n$ alternating sign matrix is an $n \times n$ matrix whose entries are $0, \pm 1$, whose row and column sums are all equal to 1, and such that the nonzero entries of every row and column alternate in sign. An element a_{ij} of a strict Gelfand pattern T is special if $2 \le i \le j \le n$ and $a_{i-1,j-1} < a_{ij} < a_{i-1,j}$. Let $s(T)$ denote the number of special elements of T. There is a simple bijection [6] between monotone triangles T of length n and alternating sign matrices A of length n, for which $s(T)$ is the number of -1's in A. There is also a simple bijection (e.g., [2]) between Gelfand patterns with first row $\lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_1$ and column-strict plane partitions of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ (i.e., λ_i parts in row i) and largest part $\le n$.

Example. The seven monotone triangles T of length 3 are given by

123	123	123	123	123	123	123
12	12	13	13	13	23	23
1	2	1	2	3	2	3

All of them satisfy $s(T) = 0$ except the fourth, for which $s(T) = 1$.

A shifted plane partition is defined analogously to plane partition, except that the array (π_{ij}) is defined only for $1 \le i \le j$. Such terminology as "column-strict" and "number of rows" is carried over in an obvious way to shifted plane partitions. For example,

```
554331
 4322
  11
```

is a column-strict shifted plane partition with 3 rows and 6 columns.

Let μ be an integer. A column-strict shifted plane partition (CSSPP) is of <u>class μ</u> if the first entry of each row exceeds the row length by precisely 2μ. There is a simple bijection [8] between CSSPP's of class 1 with $\leq n$ columns and <u>descending plane partitions</u> (as defined by G. Andrews [1]) with largest part $\leq n+1$. There is also a simple bijection between CSSPP's of class 0 with $\leq n$ columns and cyclically symmetric plane partitions with largest part $\leq n$ (see [8]). A part π_{ij} of a CSSPP of class μ is <u>special</u> if $\mu < \pi_{ij} \leq j-i+\mu$, and we write $s(T)$ for the number of special parts of T.

<u>Example</u>. The seven CSSPP's of class 1 with ≤ 2 columns are given by

$$\phi \qquad 3 \qquad 41 \qquad 42 \qquad 43 \qquad 44 \qquad \begin{array}{l}44\\ \;\,3\end{array}$$

All of these satisfy $s(T) = 0$ except the fifth, for which $s(T) = 1$.

We now are ready to list the conjectures (as of November, 1985), together with some related theorems.

<u>Theorem</u> (equivalent to [1, Thm. 7]). The number of CSSPP's of class 1 and $\leq n-1$ columns is equal to

$$A_n := \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+\cancel{1}\,i)!} .$$

<u>Conjecture 1</u> [6]. The number of $n \times n$ alternating sign matrices is equal to A_n.

<u>Conjecture 2</u> [7, Conj. 1][11, Case 10]. The number of totally symmetric $(2n,2n,2n)$-self-complementary plane partitions is equal to A_n.

solved by G. Andrews

<u>Note</u>. One can give a bijection [7] between totally symmetric $(2n, 2n,2n)$-self-complementary plane partitions and shifted plane partitions $\pi = (\pi_{ij})$ of shape $(n-1,n-2,\ldots,1)$ such that $n-\cancel{1}\,i \leq \pi_{ij} \leq n$ for all parts π_{ij} of π.

<u>Note</u>. It is not known whether the number of $n\times n$ alternating sign matrices is equal to the number of totally symmetric $(2n,2n,2n)$-self-complementary plane partitions.

<u>Conjecture 3</u> [6, Conj. 2]. The number of monotone triangles of length n with bottom entry $a_{nn} = r$ (equivalently, the number of $n\times n$ alternating sign matrices (α_{ij}) with $\alpha_{nr} = 1$) is equal to

$$\binom{2n-2}{n-1}^{-1}\binom{n+r-2}{n-1}\binom{2n-r-1}{n-1}A_{n-1}$$

Note. One easily deduces Conjecture 1 from Conjecture 3 .

Conjecture 4 [6, Conjs. 4 and 5]. Define $A_n(x) = \sum_T x^{s(T)}$, where T ranges over all monotone triangles of length n . Define $B_{2n+1}(x) = \sum_T x^{s(T)}$, where T ranges over all strict Gelfand patterns with first row 1,3,5,...,2n-1. Then there exist polynomials $B_{2n}(x)$ for which

$$A_n(x) = \begin{cases} B_n(x)B_{n+1}(x) & , n \text{ odd} \\ 2\ B_n(x)B_{n+1}(x) & , n \text{ even} . \end{cases}$$

Note. For a conjectured explicit value of $B_{2n+1}(1)$, see the note following Conjecture 9.

Note. Conjecture 1 is equivalent to the assertion $A_n(1) = A_n$. It is not difficult to show [6, Cor. on p. 358] that $A_n(2) = 2^{\binom{n}{2}}$. In fact, much more can be said concerning the weight $2^{s(T)}$ of a strict Gelfand pattern T , and there are strong connections with the theory of symmetric functions. For instance, if $\sigma_i(T)$ denotes the i-th row sum of T , then it can be shown that

$$\sum_T 2^{s(T)} x_1^{\sigma_1(T)-\sigma_2(T)} x_2^{\sigma_2(T)-\sigma_3(T)} \cdots x_n^{\sigma_n(T)}$$

$$= s_\lambda(x_1,\ldots,x_n)\prod_{1\le i<j\le n}(x_i+x_j),$$

where T ranges over all strict Gelfand patterns with first row $(\lambda_n,\lambda_{n-1}+1,\ldots,\lambda_1+n-1)$, and where s_λ denotes the Schur function (as defined, e.g., in [4] or [10]) corresponding to the partition $\lambda = (\lambda_1,\ldots,\lambda_n)$.

Conjecture 5 [6, Conj. 6]. $A_n(3) = 3^{t(n)}H_n$, where

$$t(n) = \begin{cases} m(m-1) & , n = 2m \\ m^2 & , n = 2m+1 , \end{cases}$$

and where H_n is determined by the recurrence

$$H_0 = 1, \quad \frac{H_{2n+1}}{H_{2n}} = \frac{\binom{3n}{n}}{\binom{2n}{n}}, \quad \frac{H_{2n}}{H_{2n-1}} = \frac{4}{3}\,\frac{\binom{3n}{n}}{\binom{2n}{n}}.$$

Conjecture 6 [8, Conj. in Sect. 4]. Define $Z_n(x,\mu) = \sum_T x^{s(T)}$, where T ranges over all CSSPP of class μ and rows of length $\leq n$. Then $Z_n(2,\mu)$ is determined by the recurrence $Z_1(2,\mu) = 2$,

$$\frac{Z_{2m}(2,\mu)}{Z_{2m-1}(2,\mu)} = 2^m \prod_{i=1}^{m} \frac{\mu+2m+2i-1}{m+i}$$

solved by
G. Andrews

$$\frac{Z_{2m+1}(2,\mu)}{Z_{2m}(2,\mu)} = 2^{m+1} \prod_{i=1}^{m} \frac{\mu+2m+2i-1}{m+i}.$$

Note. A strengthening of Conjecture 1 is given by $Z_n(x,1) = A_n(x)$, where $A_n(x)$ is defined in Conjecture 4 (see [8, Sect. 4]).

Conjecture 7. (see [11, Case 4]). The number of totally symmetric plane partitions with largest part $\leq n$ is equal to

$$T_n = \prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}.$$

Note. It is not hard to show that the number of totally symmetric plane partitions with largest part $\leq n$ is also equal to

a) the number of row-strict shifted plane partitions with largest part $\leq n$,

b) the number of order ideals of the poset $L(3,n)$ of Ferrers diagrams fitting in a $3\times n$ rectangle, ordered by inclusion,

c) the sum of the minors of all orders (including the void minor equal to 1) of the matrix whose (i,j)-entry is $\binom{i}{j}$ for $0\leq i, j\leq n-1$.

Note. All quantities arising in connection with Conjecture 7 have natural q-analogues. The q-analogue of T_n is

$$T_n(q) = \prod_{1\leq i\leq j\leq k\leq n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

The q-analogue of the number of totally symmetric plane partitions with largest part $\leq n$ is the polynomial $N_G'(B;q)$ defined in [11], where $B = B(n,n,n)$ and $G = S_3$. The q-analogue of (a) is just $\Sigma q^{|\pi|}$, summed over all π satisfying (a). The q-analogue of (b) is $\Sigma q^{|I|}$, summed over all order ideals I of $L(3,n)$. Finally, the q-analogue of

(c) corresponds to the matrix with (i,j)-entry $q^{i+1+\binom{j+1}{2}}\begin{bmatrix} i \\ j \end{bmatrix}$ $0 \le i, j \le n-1$. As in Conjecture 7, the last four quantities are known to be equal, and are conjectured to equal $T_n(q)$.

Conjecture 8 (D. Robbins, et al.; see [11, Case 9]). The number of cyclically symmetric $(2n,2n,2n)$-self-complementary plane partitions is equal to A_n^2. } solved by G. Kuperberg

Note. It is not known whether the number of cyclically symmetric $(2n,2n,2n)$-self-complementary plane partitions is the square of the number which are also symmetric (Conjecture 2). Perhaps there is a bijection which shows the equivalence of Conjectures 2 and 8 without proving either one.

Conjecture 9 (implicit in [8]). The number F_n of $n\times n$ alternating sign matrices which are invariant under a reflection about a vertical axis is given by the recurrence

$$F_1 = 1,\ F_{2n} = 0\ ,\ \frac{F_{2n+1}}{F_{2n-1}} = \frac{\binom{6n-2}{2n}}{2\binom{4n-1}{2n}}\ .$$

Note. It is easy to see that $F_{2n+1} = B_{2n+1}(1)$, as defined in Conjecture 4. Moreover, the number of strict Gelfand patterns (a_{ij}) with first row $1,3,\ldots 2n-1$ which are "flip-symmetric", in the sense that $a_{ij} + a_{i,n+i-j} = 2n$ for all $1 \le i \le j \le n$, is equal to P_{2n+1}, as defined in Conjecture 12.

Conjecture 10 [7, Conj. 5]. The number of $n\times n$ alternating sign matrices which are invariant under a 180° rotation is equal to the quantity H_n of Conjecture 5.

Note. It is not known whether Conjectures 5 and 10 are equivalent, i.e., whether $3^{-t(n)}A_n(3)$ is equal to the number of $n\times n$ alternating sign matrices invariant under a 180° rotation.

Conjecture 11 (D. Robbins; see [9, Sect. 3.5]). The number Q_n of $n\times n$ alternating sign matrices which are invariant under a 90° rotation is given by the recurrence

$$Q_1 = 1\ ,\ Q_{4n+2} = 0\ ,\ \frac{Q_{4n+3}}{Q_{4n+1}} = \frac{\binom{3n+1}{n}^2}{\binom{2n}{n}^2}$$

$$\frac{Q_{4n+5}}{Q_{4n+3}} = \frac{3\binom{3n+2}{n}^2}{\binom{2n+1}{n}^2}, \quad \frac{Q_{4n}}{Q_{4n-1}} = \frac{2\binom{3n-1}{n}}{\binom{2n}{n}}.$$

Conjecture 12 (W.H. Mills; see [9, Sect. 4.2]). The number P_n of $n \times n$ alternating sign matrices which are invariant under reflections in both a horizontal axis and a vertical axis is given by the recurrence $P_1 = 1$, $P_{2n} = 0$,

$$\frac{P_{4n+3}}{P_{4n+1}} = \frac{(3n+1)\binom{6n}{2n}}{(4n+1)\binom{4n}{2n}}, \quad \frac{P_{4n+1}}{P_{4n-1}} = \frac{(3n-1)\binom{6n-3}{2n-1}}{(4n-1)\binom{4n-2}{2n-1}}.$$

Conjecture 13 (D. Robbins; see [9, Sect. 3.7]). The number X_n of $n \times n$ alternating sign matrices which are invariant under reflections in both diagonals satisfies $X_1 = 1$,

$$\frac{X_{2n+1}}{X_{2n-1}} = \frac{\binom{3n}{n}}{\binom{2n-1}{n}}.$$

Note. There are no conjectures at present for the cardinalities of two additional symmetry classes of $n \times n$ alternating sign matrices, viz., those that are symmetric matrices (i.e., invariant under a reflection in the main diagonal), and those that are invariant under the full symmetry group of the square. Call these cardinalities S_n and K_n, respectively. Moreover, no conjecture is known for X_{2n} as defined by Conjecture 13.

Note. There are a total of ten symmetry classes of plane partitions with $\leq r$ rows, $\leq s$ columns, and largest part $\leq t$ [11]. Seven of these classes have been successfully counted, while the remaining three correspond to Conjectures 2, 7, and 8.

Note. In [9] many of the above conjectures related to symmetry classes of alternating sign matrices are strengthened by considering various weights on the alternating sign matrices under consideration. There also appear some surprising connections between different symmetry classes (which follow from the conjectures themselves, but which perhaps can be proved independently). For instance, it follows from Conjecture 5 above that $H_{2n} = Z_n(1,0)A_n$ (a special case of

[9, Conj. 3.3.1]), and from Conjectures 1,5, and 11 above that $Q_{4n} = A_n^2 H_{2n}$, $Q_{4n+1} = A_n^2 H_{2n+1}$, $Q_{4n-1} = A_n^2 H_{2n-1}$ (a special case of [9, Conj. 3.5.1]).

We conclude with a table listing some of the values of the functions discussed above. Many of these values are taken from [9]. An entry marked * denotes a number of eight digits or more whose value we omit.

	n	1	2	3	4	5	6	7	8	
A5130 ✓	A_n	1	2	7	42	429	7436	218348	*	A5130
A5158 ✓	H_n	1	2	3	10	25	140	588	5544	alt. sign mxs
A5157 ✓	T_n	2	5	16	66	352	2431	21760	252586	totally sym plane partitions
A5159 ✓	$Z_n(1,0)$	2	5	20	132	1452	26741	826540	*	
A5156 ✓	F_{2n-1}	1	1	3	26	646	45885	9304650	*	alternating sign mxs
A5160 ✓	Q_n	1	0	1	2	3	0	12	40	
A5161 ✓	P_{2n-1}	1	1	1	2	6	33	286	4420	
A5162 ✓	X_n	1	2	3	8	15	52	126	568	
A5163	S_n	1	2	5	16	67	368	2630	24376	
A5164	K_{2n-1}	1	1	1	2	4	13	46	248	

<u>Moreover</u>: $Q_9 = 100$, $Q_{10} = 0$, $Q_{11} = 1225$, $Q_{12} = 6860$,

$X_9 = 1782$, $X_{10} = 10436$, $X_{11} = 42471$, $X_{12} = 323144$, $X_{13} = 1706562$, $X_{14} = 16866856$

$K_{17} = 1516$

$B_1(x) = B_2(x) = B_3(x) = 1$, $B_4(x) = 6+x$, $B_5(x) = 2+x$, $B_6(x) = 60+70x+12x^2+x^3$, $B_7(x) = 6+13x+6x^2+x^3$, $B_8(x) = 840+3080x+3038x^2+1224x^3+195x^4+20x^5+x^6$, $B_9(x) = 24+136x+234x^2+176x^3+63x^4+12x^5+x^6$

$Z_1(2,\mu) = 2$, $Z_2(2,\mu) = 2(\mu+3)$, $Z_3(2,\mu) = 4(\mu+3)^2$, $Z_4(2,\mu) = \frac{4}{3}(\mu+3)^2(\mu+5)(\mu+7)$, $Z_5(2,\mu) = \frac{8}{9}(\mu+3)^2(\mu+5)^2(\mu+7)^2$.

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