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DIVISIBILITY PROPERTIES OF SOME FIBONACCI-TYPE SEQUENCES

A.F. HORADAM, R.P. LOH AND A.G. SHANNON

A generalized Fibonacci-type sequence is defined from a fourth order homogeneous linear recurrence relation, and various divisibility properties are developed. In particular, the notion of a proper divisor is modified to develop formulas for proper divisors in terms of the general terms of the recurrence sequences and various arithmetic functions.

1. THE SEQUENCE \( \{A_n(x)\} \)

The main sequence of interest is \( \{A_n(x)\} \) defined by

\[
\begin{align*}
A_0(x) &= 0, \\
A_1(x) &= 1, \\
A_2(x) &= 1, \\
A_3(x) &= x + 1 \\
A_n(x) &= xA_{n-2}(x) - A_{n-4}(x) \quad \text{for } n \geq 4.
\end{align*}
\]

(1.1)

For unrestricted \( n \), it follows from (1.1) that

\[
A_{-n}(x) = -A_n(x).
\]

(1.2)

The auxiliary equation for \( \{A_n(x)\} \) is \( x^4 - x^2 + 1 = 0 \) which has roots \( s, t \). It is given by

\[
\begin{align*}
s^2 &= \frac{1}{4}(x + \sqrt{x^2 - 4}) \\
t^2 &= \frac{1}{4}(x - \sqrt{x^2 - 4})
\end{align*}
\]

so that \( s^2 + t^2 = x, s^2t^2 = 1 \)

(1.3)

whence

\[
\begin{align*}
s &= \frac{1}{2}(\sqrt{x - 2} + \sqrt{x + 2}) \\
t &= \frac{1}{2}(\sqrt{x - 2} - \sqrt{x + 2})
\end{align*}
\]

so that \( s + t = \sqrt{x - 2}, st = -1 \)

(1.4)

From the initial conditions in (1.1) we derive

\[
A_{2n}(x) = \frac{s^{2n} - t^{2n}}{s^2 - t^2}
\]

(1.5)

Mathematical induction and the recurrence relation (1.1) lead to

\[
A_{2n+2}(x) = A_{2n+1}(x) - A_{2n}(x)
\]

(1.6)
whence, by (1.5),
\[
A_{2n+1}(x) = \frac{s^{2n}(s^2 + 1) - t^{2n}(t^2 + 1)}{s^2 - t^2}
\]

The generating function for \(\{A_n(x)\}\) is given by
\[
\sum_{n=1}^{\infty} A_n(x) t^n = \frac{t + t^2 + t^3}{1 - xt^2 + t^6}
\]

Put \(x = 3\) and let \(\alpha, \beta\) be the roots of \(x^2 - r - 1 = 0\). Then (1.3) yields
\[
s^2 = \alpha^2, t^2 = \beta^2, s^2 - t^2 = \alpha - \beta = \sqrt{5}, s^2 + 1 = \sqrt{5}a, t^2 + 1 = -\sqrt{5}b.
\]

From (1.5), (1.7) and (1.9), we have
\[
\begin{align*}
A_{2n}(3) &= \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = F_{2n} \\
A_{2n+1}(3) &= \alpha^{2n+1} + \beta^{2n+1} = L_{2n+1}
\end{align*}
\]
in which \(F_{2n}, L_{2n+1}\) are the \(2n\)-th Fibonacci number and \((2n+1)\)-th Lucas number respectively. Equation (1.6) is by (1.2) an instance of the well-known result:
\[F_{k+2} + F_k = L_{k+1}\]
which is true for all \(k\). The generating function for \(\{A_n(3)\}\) follows immediately from (1.8).

Table 1 shows the first 18 terms of \(\{A_n(3)\}\), alternately the Lucas and Fibonacci numbers \(L_1, F_2, L_3, F_4, L_5, F_6, L_7, F_8, L_9, \ldots\).

Other examples of \(\{A_n(x)\}\) include
\[
\begin{align*}
|A_{2n}(-2)| &= n \in \mathbb{N} \\
|A_{2n+1}(-2)| &= 1
\end{align*}
\]

Further information about the sequence \(\{A_n(x)\}\) may be found in Shannon, Horadam, and Loh [6] where a different notation is used.

2. **PROPER DIVISORS**

Vorob'ev [7] in the concluding chapter of his book on Fibonacci numbers refers to the notion of a *proper divisor*. We extend this idea a little as follows:

**Definition.** For any sequence \(\{u_n\}, n \geq 1\), where \(u_n \in \mathbb{Z}\) or \(u_n(x) \in \mathbb{Z}(x)\), the *proper divisor* \(w_n\) is the quantity implicitly defined, for \(n \geq 1\), by \(w_1 = u_1\) and \(w_n = \max(d: d\mid u_n \text{ and } \gcd(d, w_m) = 1 \text{ for every } m < n)\).

(Strictly speaking, the second equation is all that is necessary here, since for \(n = 1\) its \(\gcd\) condition is vacuous and so \(w_1 = u_1\) follows.)

Proper divisors \(w_n\) for the sequence of integers \(\{A_n(3)\}\) are the integers listed in Table 1. (Recall (1.10).)
Proper divisors for the sequence of polynomials $\{A_n(x)\}$ are shown in Table 2. These proper divisors $w_n(x)$ are monic polynomials (over the integers).

Table 1. Proper divisors for $\{A_n(3)\}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_n(3)$</th>
<th>$w_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>29</td>
<td>21</td>
</tr>
<tr>
<td>8</td>
<td>29</td>
<td>21</td>
</tr>
<tr>
<td>9</td>
<td>76</td>
<td>7</td>
</tr>
<tr>
<td>10</td>
<td>55</td>
<td>19</td>
</tr>
<tr>
<td>11</td>
<td>199</td>
<td>199</td>
</tr>
<tr>
<td>12</td>
<td>144</td>
<td>144</td>
</tr>
<tr>
<td>13</td>
<td>521</td>
<td>521</td>
</tr>
<tr>
<td>14</td>
<td>377</td>
<td>377</td>
</tr>
<tr>
<td>15</td>
<td>1364</td>
<td>1364</td>
</tr>
<tr>
<td>16</td>
<td>987</td>
<td>987</td>
</tr>
<tr>
<td>17</td>
<td>3571</td>
<td>3571</td>
</tr>
<tr>
<td>18</td>
<td>2584</td>
<td>2584</td>
</tr>
</tbody>
</table>

Table 2. Proper divisors for $\{A_n(x)\}$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_n(x)$</th>
<th>$w_n(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$x+1$</td>
<td>$x+1$</td>
</tr>
<tr>
<td>4</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>5</td>
<td>$x^2+x-1$</td>
<td>$x^2+x-1$</td>
</tr>
<tr>
<td>6</td>
<td>$x^2-1$</td>
<td>$x-1$</td>
</tr>
<tr>
<td>7</td>
<td>$x^3+x^2-2x-1$</td>
<td>$x^3+x^2-2x-1$</td>
</tr>
<tr>
<td>8</td>
<td>$x^3-2x$</td>
<td>$x^2-2$</td>
</tr>
<tr>
<td>9</td>
<td>$x^4+x^3-3x^2-2x+1$</td>
<td>$x^3-3x+1$</td>
</tr>
<tr>
<td>10</td>
<td>$x^4-3x^2+1$</td>
<td>$x^2-x-1$</td>
</tr>
<tr>
<td>11</td>
<td>$x^5+x^4-4x^3-3x^2+3x+1$</td>
<td>$x^5+x^4-4x^3-3x^2+3x+1$</td>
</tr>
<tr>
<td>12</td>
<td>$x^5-4x^3+3x$</td>
<td>$x^2-3$</td>
</tr>
</tbody>
</table>

From the definition of proper divisors we have that for $\{A_n(x)\}$:

$$A_p(x) = w_p(x) w_1(x)$$

$$A_{2p}(x) = w_{2p}(x) w_p(x) w_1(x)$$

$$A_{3p}(x) = w_{3p}(x) w_p(x) w_3(x) w_1(x)$$

... 

in which $w_2(x) = w_1(x) = 1$ for notational convenience.

Hence,

$$A_n(x) = \prod_{d|n} w_d(x)$$

Theorem 1. $w_n(x) = \prod_{d|n} (A_d(x))^{\mu(n/d)}$ where $\mu$ is the Möbius function.

Proof. Taking logarithms in (2.1) we obtain

$$\ln A_n(x) = \sum_{d|n} \ln w_d(x)$$

which becomes, with the Möbius inversion formula,

$$\ln w_n(x) = \sum_{d|n} \mu(n/d) \ln A_d(x)$$
\[ w_n(x) = \prod_{d \mid n} (A_d(x))^\mu(n/d) \text{ as required.} \]

As an example,
\[ w_{12}(x) = (A_3(x))^\mu(4) \cdot (A_4(x))^\mu(3) \cdot (A_6(x))^\mu(2) \cdot (A_{12}(x))^\mu(1) \]
\[ = (x + 1)^0 x^{-1}(x^2 - 1)^{-1}(x^5 - 4x^3 + 3x)^1 \]
\[ = x^2 - 3. \]

Another approach to Theorem 1 is through cyclotomic polynomials.

3. PROPERTIES OF PROPER DIVISORS

Let \( n = \prod_{i=1}^{m} p_i^{a_i} \) where \( p_i \) are distinct primes, and let \( \nu(n) = \sum_{i=1}^{m} a_i \) be the number of prime factors of \( n \), counted with multiplicity. Further, let \( \epsilon(n) = (-1)^\nu(n) \).

Then

**Theorem 2.** \( w_n(x) = \prod_{d \in S_1} A_d(x)^{\epsilon(n)} \), where the sets \( S_1 \) comprise all positive divisors \( d \) of \( n \) such that \( n/d \) is squarefree, and \( \nu(d) \equiv i \mod(2) \) for \( i = 0, 1 \).

**Proof.** Write \( d = \prod_{i=1}^{m} p_i^{a_i} \) \((\beta_i \leq a_i)\).

Then
\[ \frac{n}{d} = \prod_{i=1}^{m} p_i^{a_i - \beta_i}, \]
and
\[ \mu(n/d) = (-1)^{\sum{(a_i - \beta_i)}} \]
\[ (0 \leq a_i - \beta_i \leq 1), \]

since \( \mu(n/d) = 0 \) if \( a_i - \beta_i \geq 2 \).

From Theorem 1 we have, using the specified notation,
\[ w_n(x) = \prod_{d \mid n} (A_d(x))^{\epsilon(n)} \cdot (-1)^{\sum{a_i - \beta_i}} \]
\[ = \prod_{d \mid n} (A_d(x))^\nu(d) \cdot (-1)^{\sum{\beta_i}} \]
\[ = \prod_{d \mid n} (A_d(x))^{\epsilon(n) \cdot \nu(d)} \]
\[ = \left( \prod_{d \in S_0} A_d(x)^{\epsilon(n)} \right) \cdot \left( \prod_{d \in S_1} A_d(x)^{\epsilon(n)} \right) \]

since \( \epsilon(n) \) will be positive or negative according as \( d \in S_0 \) or \( d \in S_1 \), respectively.
For example, \[ w_{60}(x) = \frac{A_4(x) A_6(x) A_{10}(x) A_{60}(x)}{A_2(x) A_1(x) A_{20}(x) A_{30}(x)} \]

since \[ v(60) = 4 \quad (\text{i.e. } e(60) = 1) \]
with \[ S_0 = \{4, 6, 10, 60\} \]
and \[ S_1 = \{2, 12, 20, 30\} \].

Note that \[ w(60/d) = 0 \] for \( d = 3, 5, 15 \).

**Corollary 1.** \[ w_{p^n}(x) = \frac{A_{p^n}(x)}{A_{p^n-1}(x)} \].

**Corollary 2.** \[ \frac{w_{2^k}(x) - w_{2^{k+1}}(x)}{2} = 2 \quad (k \geq 2) \].

The proof of Corollary 2 uses Corollary 1 and (1.5).

Corollary 2 may be illustrated by choosing \( k = 2 \), whence, by Table 2,

\[ w_4(x) - w_0(x) = x^2 - (x^2 - 2) = 2. \]

Other results, such as

**Corollary 3.** \[ \frac{w_{2^{k+1}}(x) - w_{2^{k+6}}(x)}{2} = 3 \]

can be similarly proved. Putting \( k = 1 \) in Corollary 3 and using Table 2, we see that

\[ w_4(x) - w_1(x) = x^2 - (x^2 - 3) = 3. \]

4. **GENERALIZED PELL NUMBERS**

Consider the forward shift operator \( E \):

\[ (E - s^2)(E - t^2) A_n(x) = 0 \]

Then (1.1), along with (1.3), can be written as

\[ (E - s)(E - t)(E + s)(E + t) A_n(x) = 0 \]

so

\[ (E - s)(E - t) \phi_n(x) = 0 \]

where

\[ \phi_n(x) = (E^2 + (s + t)E + st) A_n(x) \]

i.e.

\[ \phi_n(x) = A_{n+2}(x) + M A_{n+1}(x) - A_n(x) \]

where, by (1.4),

\[ M = s + t = \sqrt{x-2} \].

Equation (4.2) can, with (1.4), be rewritten as

\[ \phi_{n+2}(x) = M \phi_{n+1}(x) + \phi_n(x) \]

(4.4)
which is the usual form of the Pellian recurrence relation. Consequently, we may call
\( \{\phi_n(x)\} \) a generalized Pell sequence and \( \phi_n(x) \) generalized Pell numbers.

Equation (4.3) relates generalized Pell numbers to numbers of the sequence \( \{A_n(x)\} \).

When \( M = 2 \) in (4.4) (i.e., \( x = 6, s = 1 + \sqrt{2}, t = 1 - \sqrt{2} \) in (1.4)), we have
the most common form which is used to generate the ordinary Pell sequence of numbers
\( \{P_n\} = \{1, 2, 5, 12, 29, 70, 169, 408, \ldots\}, \) \( n \geq 1 \), defined (Horadam [5]) by
\[
\begin{align*}
P_{n+2} &= 2P_{n+1} + P_n \quad \text{with} \quad P_0 = 0, \quad P_1 = 1. \\
\end{align*}
\]
Furthermore, the explicit form of \( P_n \) is
\[
P_n = \frac{s^n - t^n}{s - t} \quad (s - t = 2\sqrt{2} \text{ from (1.4)}).
\]
For unrestricted \( n \), (4.6) yields
\[
P_{-n} = (-1)^{n+1} P_n.
\]
Elements of the sequences \( \{A_n(x)\}, \{\phi_n(x)\} \) and \( \{P_n\} \) are related thus, as may
be demonstrated:
\[
\begin{align*}
\phi_{2n+1}(x) &= P_{2n+1} + A_{2n+3}(x) \\
\phi_{2n}(x) &= P_{2n} - 2P_{2n+2} + A_{2n+3}(x).
\end{align*}
\]
Table 3 shows the first few numbers in the sequences \( \{A_n(6)\} \) and \( \{\phi_n(6)\} \) which
are obtained from recurrence relations (1.1) and (4.3), and are confirmed by the
recurrence relation (4.4).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_n(6) )</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>6</td>
<td>41</td>
<td>35</td>
<td>239</td>
</tr>
<tr>
<td>( \phi_n(6) )</td>
<td>8</td>
<td>19</td>
<td>46</td>
<td>111</td>
<td>268</td>
<td>647</td>
<td>1562</td>
</tr>
</tbody>
</table>

Table 3. \( A_n(6) \) and \( \phi_n(6) \)

Observe that \( \phi_0(6) = P_0 - 2P_2 + A_3(6) = 3 \) by (4.5), (4.8) and Table 3.

Values of \( \phi_{-n}(6) \) may by calculated from (4.8) in conjunction with (1.1), (1.2),
(4.7) and (4.8).

**Theorem 3.** If, in (4.4), \( M = 2N \) (even), \( n \geq 0, \phi_0(N) = 1, \phi_1(N) = N, \) then

\[
(\phi_{2n}(N) - 1)/(N^2 + 1) \quad \text{is a perfect square.}
\]

**Proof.** The explicit form of \( \phi_n(N) \) is, by the usual method,
\[
\phi_n(N) = h(c^n + d^n)
\]
where \( c = N + \sqrt{N^2 + 1} \) and \( d = N - \sqrt{N^2 + 1} \), so \( c - d = 2\sqrt{N^2 + 1} \) and \( cd = -1. \)
Hence
\[
\phi_{2n}^2(N) - 1 = \frac{(c^{2n} + d^{2n})^2 - 4}{4(N^2 + 1)} = \left(\frac{c^{2n} - d^{2n}}{c - d}\right)^2
\]
and \((c^{2n} - d^{2n})/(c - d)\) is an integer for non-negative integer \(n\).

For example
\[
\phi_4(N) - 1 = (4N(2N^2 + 1))^2 \quad \text{when} \quad n = 2.
\]

When \(N = 1\) in Theorem 3, we have the sequence \(\{\phi_n\}\) say:
\[
1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, \ldots
\]
whence
\[
\begin{align*}
\phi_{2n+1} &= A_{2n+1}(6) \\
\phi_{2n} &= A_{2n}(6)
\end{align*}
\]
and
\[
\frac{\phi_{2n}^2 - 1}{2} = \frac{2}{2n} \quad \text{(by (4.6) since \(c = s, d = t\) when \(N = 1\)).}
\]

In the illustrative example above, when \(n = 2\) we have
\[
\frac{\phi_{2}^2 - 1}{2} = \frac{289 - 1}{2} = 144 = 12^2 = p_4^2.
\]

5. A FIBONACCI-TYPE SEQUENCE

Consider the sequence \(\{Q_n(N)\}\) where \(N \neq 2\) is an integer:
\[
\begin{align*}
Q_{n+2}(N) &= N Q_{n+1}(N) - Q_n(N) \quad (n \geq 1) \\
Q_1(N) &= Q_2(N) = 1
\end{align*}
\]

The first few numbers of this sequence are given in Table 4:

<table>
<thead>
<tr>
<th>(n)</th>
<th>(Q_n(N))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3 - 1</td>
</tr>
<tr>
<td>4</td>
<td>3N - N - 1</td>
</tr>
<tr>
<td>5</td>
<td>N^3 - N^2 - 2N + 1</td>
</tr>
<tr>
<td>6</td>
<td>N^4 - N^3 - 2N^2 + 3N + 1</td>
</tr>
<tr>
<td>7</td>
<td>N^5 - N^4 - 3N^3 + 5N^2 + 3N - 1</td>
</tr>
<tr>
<td>8</td>
<td>N^6 - N^5 - 4N^4 + 7N^3 + 6N^2 - 3N - 1</td>
</tr>
<tr>
<td>9</td>
<td>N^7 - N^6 - 6N^5 + 10N^4 + 5N^3 - 6N^2 - 4N + 1</td>
</tr>
</tbody>
</table>

Table 4.
An interesting factorization result arises from these numbers, namely:

Theorem 4. \( Q_{n+1}(N) - 1 = (N - 2) A_n(N) A_{n-1}(N) \) \( (n \geq 2, N \neq 2) \).

Proof. We use induction. The result is obvious when \( n = 2, 3 \). Assume the result is true for \( n = 4, 5, \ldots, k+1 \). Then

\[
Q_{k+2}(N) = N Q_{k+1}(N) - Q_k(N)
\]

by (5.1)

\[
= N Q_{k+1}(N) - N Q_{k-1}(N) + Q_{k-2}(N)
\]

by (5.1)

\[
= N(N - 2) A_k(N) A_{k-1}(N) + N - N(N - 2) A_{k-2}(N) A_{k-3}(N)
\]

\[+ N + (N - 2) A_{k-3}(N) A_{k-4}(N) + 1 \quad \text{by hypothesis}
\]

\[= N(N - 2) A_k(N) A_{k-1}(N) + (N - 2) A_{k-3}(N) (A_{k-4}(N) - N A_{k-2}(N)) + 1
\]

\[= N(N - 2) A_k(N) A_{k-1}(N) - (N - 2) A_{k-3}(N) A_k(N) + 1 \quad \text{by (1.1)}
\]

\[= (N - 2) A_k(N) \left( N A_{k-1}(N) - A_{k-3}(N) \right) + 1 \quad \text{by (1.1)}
\]

\[= (N - 2) A_k(N) A_{k+1}(N) + 1 \quad \text{as required.}
\]

For example, \( n = 7 \) in Theorem 4 gives, with the help of Table 2,

\[Q_8(N) - 1 = (N - 2) A_7(N) A_6(N) = (N - 2)(N^2 - 1)(N^3 + N^2 - 2N - 1).
\]

Further, \( N = 3 \) in this example yields, with (1.10),

\[F_{13} - 1 = L_7 F_6 \quad (= 232 = 29 \times 8),
\]

since the numbers \( Q_n(3) \) are certain Fibonacci numbers.

Induction also leads to the result

\[
Q_n(N) = \sum_{j=0}^{[(n-2)/2]} (-1)^j \binom{n-j-2}{j} N^{n-2j-2} - \sum_{j=0}^{[(n-3)/2]} \binom{n-j-3}{j} N^{n-2j-3}
\]

(5.2)

which is a generalization of (2.8) of Barakat [1].

When \( N = 3 \), (5.2) reduces to

\[Q_n(3) = F_{2n-2} - F_{2n-4}.
\]

For example, \( Q_5(3) = 13 = F_8 - F_6 \quad (= 21 - 8) \).

A basic relationship amongst the elements of \( \{Q_n(N)\} \) is

\[Q_{n+1}(N) Q_{n-1}(N) - Q_n^2(N) = N - 2
\]

(5.3)

which is a particular case of the general result for the sequence \( \{w_n(a,b;p,q)\} \) in Horadam [4] where \( a = 1, b = 1, p = N, q = 1 \). (Equation (5.3) is the analogue for \( \{Q_n(N)\} \) of the well-known Simson result for Fibonacci numbers: \( F_{n+1} F_{n-1} - F_n^2 = (-1)^n \).
n \geq 1.

Following the approach of Hoggatt and Bicknell [3] for Fibonacci polynomials and letting $e^z$ and $e^{-z}$ be the roots of the auxiliary equation $x^2 - N r + 1 = 0$ associated with the recurrence relation (5.1), we obtain

\begin{equation}
Q_{n+1}(N) = \frac{e^{n(2n-1)z}}{e^{nz}}
\end{equation}

where $2 \cosh z = N$, $2 \sinh z = \sqrt{N^2 - 4}$.

Clearly, further results may be developed involving the sequences under consideration, e.g.

\begin{equation}
Q_n(6) = P_{2n-3} \quad (n \geq 2),
\end{equation}

and

\begin{equation}
Q'_n(6) = P_{2n-2} \quad (n \geq 1)
\end{equation}

if we define $Q'_n(N)$ as for $Q_n(N)$ in (5.1) but with the initial conditions $Q'_1(N) = 0$, $Q'_2(N) = 2$.

The theory for $Q_n(N)$ and $Q'_n(N)$ extends to negative values of $n$.

6. CONCLUSION

It is of interest to note ways in which this work can be further extended. When $N = 2$, the Pell equation can be readily related to the Diophantine equations

\[ x^2 - 2y^2 = \pm 1 \]

because of the simple continued fraction expansion of $\sqrt{2}$, namely,

\[ \sqrt{2} = [1, 2, 2, \ldots] \]

Bernstein [2] has shown how it can be further developed by considering the surd

\[ \sqrt{m} = [b_0, b_1, b_2, \ldots, b_{n-1}, 2b_0] \]

and the recurrence relation

\[ P_{j+2} = b_j P_{j+1} + P_j \quad (j > 0), \]

where $b_j$ is the $j$th partial quotient of the continued fraction, and with suitable initial conditions. Bernstein has generalized this further by using the Jacobi-Perron algorithm to accommodate linear recurrence relations of order higher than two.

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