Automatic Enumeration of Generalized Ménage Numbers

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In fond memory of Alain Lascoux (1944-2013), one of the most CREATIVE and

ORIGINAL and INTERESTING mathematicians that I have ever known

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In Sept. 15-18, I gave three talks at the 71st session of the famous Séminaire Lotharingien de Combinatoire, on "Sieve Methods in Number Theory and Combinatorics". Since I am famously against laptop talks, I had to use the tiny whiteboard in the lecture room (in the magnificient former Bishop's castle of Bertinoro). But the first day was so beautiful, that I decided to take my "class" outside, and the whole talk was entirely oral, telling the history of sieve methods in number theory, from Eratosthenes, via Viggo Brun, all the way to the wonderful Cinderalla story of Yitang Zhang.

The two other talks were indoors. At the second talk I mentioned and briefly sketched, the Brydges-Spencer Lace Expansion, that

lead, way back in the early-1990s, to the seminal work of Takashi Hara and Gordon Slade about the asymptotic behavior of the enumerating sequence of self-avoiding walks in dimensions five and up. The last talk, that formed the basis of the present article, was about counting restricted permutations via rook polynomials, and how computers can be taught to generate, in a few seconds, deep theorems, that took such great minds like Arthur Cayley, F.R.S., Sir Thomas Muir, Monsieur le colonel Charles Moreau (a decorated soldier, brilliant amateur mathematician, but not quite as good chess player), notable politician Charles-Ange Laisant, the great Major Percy MacMahon, Jacques Touchard, John Riordan (the master of ars combinatorica), the great algebraist Irving Kaplansky, the great enumerator Earl Glenn Whitehad, and numerous others (including algebaric combinatorics guru Richard Stanley, that covered rook

polynomials in his classic EC1, but regretfully did not even mention John Riordan, his analog a generation earlier).

The venue was especially auspicious, as Bertinoro is the birthplace of one of the greatest medieval rabbis, Obadiah Bertinura, the great commentator of the Mishna. My good friend Omar Foda, who participated in the conference, found his house and kindly took a picture of me standing in front Obadiah's house, and another picture of my wife Jane and I standing in front of it

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Maple Package

- **MENAGES, the MAIN Maple package**

- **BALTIC, a small package that uses the C-finite ansatz, to find empirically-yet-rigorously generating functions enumerating permutations that do not go very far, in the style of D.H. Lehmer and Vladimir Baltic.**

Sample Input and Output for MEANGES

- **To see the generating functions for rook polynomials for k-discorcant permutations for k from 1 to 4, reproducing in 12 seconds, the labor of Euler (k=1), Lucas (and Laisant, Moreau, Touchard, Kaplasnky and many other smart people) (k=2), Riordan (k=3, plus**

we got a brand-new recurrence for the enumerating sequence itself), and Whitehead(k=4). It also gives you the first 40 terms, and the 300-th term of the enumerating sequences.

the input file yields the output file.

- **It is amazing that a human (Earl Glen Whitehead) could do, by hand, the case k=4, but I am willing to bet that even he would give up for k=5, and k=6. If you want the generating functions for rook polynomials for k-discordant permutations for k from 1 to 6, (and the first 40 terms, and the 300-th term of the enumerating sequences)**

  the input file yields the output file.

- **To see a webbook that gives you the generating functions for rook polynomials for enumerating permutations π of**

{0,1,..,n-1} such that π[i]-i mod n is never in the set S, for all subsets S of {0,1,2,3,4} including 0, and in many cases, nice recurrences for the enumerating sequences themselves, and the first 40 terms, and the 400-th term

the input file yields the output file.

- **To see a webbook that gives you the generating functions for rook polynomials for enumerating permutations π of {1,..,n} such that π[i]-i is never in the set S, for all subsets S of {-2,-1,0,1,2} with at least two elementes, and in many cases, nice recurrences for the enumerating sequence itself,**

  the input file yields the output file.

- **To see the first 30 terms, linear recurrences, asymptotics, and the 1000-th term of the sequences "number of (usual)**

permutations π of {1,...n}" where π_i -i is never in {0} (the good-old derangements) and, when π_i -i is never in {0,1} (the straight Menages problem) [Using the "clever" approach inpspired by Kaplansky's solution]

the input file yields the output file.

Here we find A000166 (the famous derangements number) and A000271, the straight Ménage numbers.

- **To see the first 30 terms, linear recurrences, asymptotics, and the 1000-th term of the sequences "number of (usual) permutations π of {1,...n}" where π_i -i is never in {0} (the good-old derangements) and, when π_i -i is never in {0,1} (the straight Menages problem) [Using the "clever" approach inpspired by Kaplansky's solution] (so far it is the same**

as above), as well as for the enunerating sequence of permutations π for which π_i -i is never in {0,2} and the enumerating sequence of permutations where π_i -i is never in {0,1,2} (that took the great John Riordan many a long months (by hand!))

the input file yields the output file.

[Note that the sequence for S={0,2} is not yet (Dec. 26, 2013) in the OEIS. The sequence for S={0,1,2} is A0001887, considered by John Riordan in 1963. Note that Shalosh redid in less than three seconds what took Riordan probably a couple of months, and it did much more, and found a seventh-order linear recurrence equation with polynomial coefficients (at worst quadratic) satisfied by the sequence, as well as asymptotics to order 10.]

- **To see the more extensive date for all sequences enumerating permutations π for which π_i-i is NEVER in a given, prescribed set S, for ALL subsets of {0,1,2,3} containing 0, [still using the "clever" approach inspired by Kaplansky's solution]**

  the input file yields the output file.

  [Note that for the sets {0,1,3}, {0,2,3}, and {0,1,2,3} there is no linear recurrence of "complexity" ≤ 20, so it only returned the first 30 terms of the enumerating sequece. One can increase the the third argument of pocedure Sefer to get them, if one wishes. Many of these sequences are not yet (Dec. 26, 2013) in OEIS]

- **To see the first 30 terms, a recurrence, asymptotics, and the 1000-th term for ALL sequences π for which π_i-i is NEVER**

in a given, prescribed set S, for ALL 31 non-empty subsets of {-2,-1,0,1,2} [Using the "empircal-yet-rigorous" Zeilberger-style approach]

the input file yields the output file.

[Note, some of the theorems are trivially equivalent to each other, of course, but who cares? Many of these sequences are not yet (Dec. 26, 2013) in OEIS.]

- **Suppose there are n diners sitting around a ROUND table. After the main course is over, they all go for a little walk in the woods, and then return for dessert. In how many ways can they be reseated in such a way that the "distance" (by number of chairs), counted clockwise, between the location of each diner during the main course and during the dessert is NEVER in a prescribed set S. Let's call this set $a_S(n)$.**

[Note that when S={0} it is derangements, and S={0,1} it is , A0000179, the classical problème des ménages]

If you want to see interesting information about these sequences for ALL 7 non-empty subsets, S, of {0,1,2,3} containing 0, the

the input file yields the output file.

[It also uses the "empircal-yet-rigorous" Zeilberger-style approach]
For S={0,1,2,3} is is A004307
For S={0,2,3} it is not yet (Dec. 26, 2013) in OEIS.
For S={0,1,3} it is not yet (Dec. 26, 2013) in OEIS.
For S={0,1,2} (alias S={-1,0,1}) it is A000183 that goes back to John Riordan (1954), but the recurrence seems to be new.
For S={0,3} it is not yet (Dec. 26, 2013) in

OEIS. (Note the complicated recurrence) For S={0,2} (alias S={-1,1}) it is not yet (Dec. 26, 2013) in OEIS! This is the numbers of ways of reseating n people around a round table where it is OK to go back to the original chair, but you can't seat in an adjacent chair. Note the elegant recurrence
$a(n) = na(n-1)+3a(n-2)+(-2n+6)*a(n-3)-3a(n-4)+(n-6)a(n-5)+a(n-6)$.
In the old days, it would have been worthy of a whole paper!

- **To see, yet another time, the good old derangement numbers, i.e. the number of ways of reseating n diners in a round table, where no one can go back to the original chair (of course, it is the same whether the table is circular or straight).**

  the input file yields the output file.

- **To see, yet another time, the good old Ménage numbers, A0000179, i.e. the number of ways of reseating n diners in a round table, where no one can go back to the original chair, and the chair next to it (going clockwise),**
  the input file yields the output file.

- **To see, yet another time, information about the sequence enumerating the number of ways of reseating n diners in a round table, where each diner must avoid the original chair and the two chairs next to it, i.e. sequence A000183 that goes back to John Riordan (1954),**

  the input file yields the output file.

  [Note the complicated (new!) 8-th order linear recurrence with quartic coefficients, and the implied asymptotics.]

- **To see information about the sequence enumerating the number of ways of reseating n diners in a round table, where each diner must move at least four chairs clockwise (i.e. S={0,1,2,3}), a sequence considered by "Earl Glen Whitehead, Jr., Four-discordant permutations, J. Austral. Math. Soc. Ser. A 28 (1979), no. 3, 369-377." i.e. sequence [A004307](A004307) ,**

  the input file yields the output file.

  [Note the complicated (new!) 21-th order linear recurrence with degree-eight coefficients!]

- **To see information about the sequence enumerating the number of ways of reseating n diners in a round table, where each diner must move at least five chairs clockwise (i.e. S={0,1,2,3,4}), i.e. sequence**

A189389,
the input file yields the output file.
[Shalosh refused to find a recurrence, to save bytes]

- **To see information about the sequence enumerating the number of ways of reseating n diners in a round table, where each diner must move at least six chairs clockwise (i.e. S={0,1,2,3,4,5}), i.e. sequence A184965,**

  the input file yields the output file.
  [Shalosh refused to find a recurrence, to save bytes]

Sample input and output for the Maple package BALTIC

- **To see Lehmer-Stanley-Baltic automatically (and rigorously) generated rational functions expressions, (if there is enough data to guess it (rigorously!) with M data points), and, for the sake of Sloane, the first 30 terms of the sequence, for the generating functions of STRAIGHT (usual) permutations π of length n with the condition that π_i -i MUST belong to a preassigned set S of the form {a,a+1, ...a+b} for -N ≤ a ≤ -2 < b ≤ N then**

    - **N=4, M=40**
      input file yields the output file.
    - **N=5, M=40**
      input file yields the output file.
    - **N=5, M=50**
      input file yields the output file.

- **To see Lehmer-Stanley-Baltic automatically (and rigorously) generated rational functions expressions, (if there is**

enough data to guess it (rigorously!) with M data points), and, for the sake of Sloane, the first 20 terms of the sequence, for the generating functions of CIRCULAR permutations π of length n with the condition that π_i -i MUST belong to a preassigned set S of the form {a,a+1, ...a+b} (i.e. generating functions of sequences of circulant permanents) for -N ≤ a ≤ -2 < b ≤ N then

- **N=2, M=35**
  input file yields the output file.
- **N=3, M=30**
  input file yields the output file.
- **N=3, M=40**
  input file yields the output file.

Personal Journal of Shalosh B. Ekhad and Doron Zeilberger

Doron Zeilberger's Home Page