Take any \( r \geq 1 \). Note that \( \varphi(p^r) = (p - 1)p^{r-1} \), where \( \varphi \) is A000010 and \( p \) is prime.

**Theorem 1** If \( p \) is a member of A003629, i.e. an odd prime for which 2 is not a square, then

\[
2^{\varphi(p^r)/2} \equiv -1 \pmod{p^r}
\]

**Proof** Let \( t \) be a primitive root mod \( p^r \). Then \( 2 \equiv t^k \pmod{p^r} \) for some \( k \) with \( 1 \leq k < \varphi(p^r) \), and 2 is a square mod \( p \) iff 2 is a square mod \( p^r \) iff \( k \) is even. Since this is not the case, \( k \varphi(p^r)/2 \) is not divisible by \( \varphi(p^r) \), so \( 2^{\varphi(p^r)/2} \equiv t^{k\varphi(p^r)/2} \neq 1 \pmod{p^r} \). On the other hand, letting \( x = 2^{\varphi(p^r)/2} \) we have \( x^2 \equiv 1 \pmod{p^r} \), and since \( x^2 - 1 = (x - 1)(x + 1) \) and only one of these can be divisible by \( p \), that implies \( x \equiv -1 \pmod{p^r} \).