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Entringer
THE NUMBER OF COPRIME CHAINS WITH LARGEST MEMBER $n$

R. C. ENTRINGER

1. In a previous paper [1] a coprime chain was defined to be an increasing sequence \( \{a_1, \ldots, a_k\} \) of integers greater than 1 which contains exactly one multiple of each prime equal to or less than \( a_k \).

We let \( s(n), n > 1 \), denote the number of coprime chains with largest member \( n \). For convenience we define \( s(1) = 1 \).

In this paper we will obtain a partial recursion formula for \( s(n) \) and an asymptotic formula for \( \log s(n) \). A table of values of \( s(n), n \leq 113 \), is also provided.

In the following \( p \) will designate a prime and \( p_i \) will designate the \( i \)th prime.

2. **Lemma 1.** \( A = \{a_1, \ldots, a_k \neq 2\} \) is a coprime chain iff
   
   (i) \( A' = \{a_1, \ldots, a_{k-1}\} \) is a coprime chain,
   
   (ii) \( p_{i-1} \) is the largest prime in \( A' \).

**Proof.** If \( A = \{a_1, \ldots, a_k \neq 2\} \) is a coprime chain, then

(ii) \( p_{i-1} \) is in \( A \) (and therefore is the largest prime in \( A' \)) since by Bertrand's Postulate \( 2p_{i-1} > p_i \), and

(iii) If \( A' \) is not a coprime chain, then there is a prime \( p \leq a_{k-1} \) dividing no member of \( A' \). Thus \( p \) divides (and therefore is equal to) \( a_k \) since \( A \) is a coprime chain, but this is impossible since \( a_{k-1} < a_k \).

To prove the converse we note that if \( A \) is not a coprime chain, then \( p_i \) divides some member of \( A' \) and therefore \( p_{i-1} < a_{k-1}/2 \). But again by Bertrand's Postulate there is a prime between \( a_{k-1}/2 \) and \( a_k \) occurring in \( A' \) which contradicts (ii).

A direct result of this lemma is:

**Theorem 2.** \( s(p_i) = \sum_{n=p_{i-1}}^{p_i-1} s(n), i \geq 2. \)

**Theorem 3.** \( s(p) = \sum_{n<p} s(n) \) (\( n \) not prime).

**Proof.** The assertion holds for \( p = 2 \). Now let \( q \) and \( p \) be successive primes with \( q < p \). If \( s(q) = \sum_{n<q} s(n) \) (\( n \) not prime), then

\[
s(p) = s(q) + \sum_{q<n<p} s(n) = \sum_{n<p} s(n) \quad (n \text{ not prime})
\]

Received by the editors April 27, 1964.

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then we have a contradiction, while \( yA = (0) \) implies (\( A \) being simple) that \( y = 0 \), which also is a contradiction. Thus we have shown \([U, U] \subset Z\).

This result indeed generalizes the work of [4].

**Theorem 4.** If \( A \) is simple (then \([A, A] = A\)) and \( U \) is a proper Lie ideal of \([A, A]\), then \( U \) is contained in the center of \( A \) except where \( A \) is of characteristic 2 and 4-dimensional over \( Z \), a field of characteristic 2.

**Proof.** Define \([U, U] = U^{(1)}\) and \( U^{(n+1)} = [U^{(n)}, U^{(n)}]\) for all \( n \geq 1 \). Then, since \( A \) is simple, it has no nonzero nilpotent ideals. Thus, except in characteristic 2, \([U, U] \subset Z\) or \( U = A. \) If the former, then Theorems 7 and 9 of [4], in the case not characteristic 3, and Lemma 3 of [1] in this case implies \( U \subset Z. \) Now, by these same results, if \( U^{(2)} \subset Z, \) then \( U \subset Z. \) Hence \([U^{(2)}, U] = A. \) Thus, by Lemma 9 of [2] we have \([U^{(2)}, A] = [A, A]\), which contradicts \( U \) being proper. Lemma 1 of [1] yields the result when \( A \) is of characteristic 2.

The author wishes to express his thanks to the referee, I. N. Herstein, for his suggestions.

**References**


5. ———, *Topics in ring theory*, Univ. of Chicago, Chicago, Ill., 1965.


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by Theorem 2 and the theorem follows by induction.

3. The above result indicates marked irregularities in $s(n)$, however, we can approximate $\log s(n)$ asymptotically.

Theorem 4. $\log s(n) \sim \sqrt{n}$.

Proof. Every coprime chain $A(n)$ can be constructed in the following manner. Let $q_i$, $i=1, \ldots, k$, $q_i > q_{i+1}$ for $i < j$ be those primes less than $\sqrt{n}$ and not dividing $n$. Choose any multiple $m_1q_1$ of $q_1$ so that $m_1q_1 \leq n$ and $(m_1, n) = 1$. If $q_2 \mid m_1$ let $m_2 = 0$. If $q_2 \not\mid m_1$, choose any multiple $m_2q_2$ of $q_2$ so that $m_2q_2 \leq n$ and $(m_2, nm_1q_1) = 1$. This process is continued by choosing $m_i = 0$ if $q_i \mid m_j$ for some $j = 1, \ldots, i - 1$, otherwise choosing any multiple $m_iq_i$ of $q_i$ so that $m_iq_i \leq n$, $(m_i, nm_1q_1 \cdots m_{i-1}q_{i-1}) = 1$. The set $\{m_1q_1, \ldots, m_kq_k\} - \{0\}$ can then be extended to a coprime chain by appending $n$ and those primes $p$ between $\sqrt{n}$ and $n$ which do not divide $n$ or any $m_i$, and reordering if necessary. This extension is unique since any multiple of a prime $p$, other than $p$ itself, must either be larger than $n$, not relatively prime to $n$, or not relatively prime to all $m_iq_i$. Therefore

$$\log s(n) \leq \log \left[ \frac{n}{p} \right] \leq \sum_{p \leq \sqrt{n}} \log n - \sum_{p \leq \sqrt{n}} \log p = \{1 + o(1)\} \sqrt{n}.$$

To obtain a lower bound for $\log s(n)$, coprime chains are constructed by choosing the $m_i$ in the following manner. Let $m_1$ be 1 or any prime satisfying $\sqrt{n} < m_1 \leq n/q_1$, $m_1 \mid n$. There are at least $\pi(n/q_1) - \pi(\sqrt{n}) - 1$ choices for $m_1$ since there is at most one prime in the given range which divides $n$. Let $m_2$, be 1 or any prime satisfying $\sqrt{n} < m_2 \leq n/q_2$, $m_2 \mid nm_1$. There are at least $\pi(n/q_2) - \pi(\sqrt{n}) - 2$ choices for $m_2$. This process is continued until all multiples $m_iq_i$ have been chosen. In general there are at least

$$\pi\left(\frac{n}{q_i}\right) - \pi(\sqrt{n}) - i \geq \pi\left(\frac{n}{q_i}\right) - \pi(\sqrt{n}) - \{\pi(\sqrt{n}) - \pi(q_i)\}$$

$$= \pi\left(\frac{n}{q_i}\right) - 2\pi(\sqrt{n}) + \pi(q_i)$$

choices for $m_i$. The set $\{m_1q_1, \ldots, m_kq_k\}$ is then extended to a coprime chain as previously indicated. If $\pi(n/q_i) - 2\pi(\sqrt{n}) + \pi(q_i) \leq 0$, then $m_i$ is chosen to be 1; hence the above construction is valid.

In the remainder of the proof we assume $\epsilon$ given such that $0 < \epsilon < 1/2$. Define $\delta$ by $n^\epsilon/\delta = 2(1 - \epsilon) \sqrt{n}$, $1/\log n < \delta < 1/2$. Then using certain results from [2] we have
\[
\log s(n) \geq \sum_{p \leq \sqrt{n}; \ p \nmid n} \log \left\{ \pi \left( \frac{n}{p} \right) - 2\pi(\sqrt{n}) + \pi(p) \right\}
\]

\[
\Rightarrow \sum_{n \leq \sqrt{n}; \ p \nmid n} \log \left\{ \frac{n}{p} - \frac{4\sqrt{n}}{\log n - 3} + \frac{p}{\log p} \right\} - \sum_{p | n} \log 2n
\]

\[
= \sum_{p \leq \sqrt{n}} \log \left( \frac{n}{p} \right) - \frac{4\sqrt{n}}{\log n - 3} + \frac{p}{\log p}
\]

\[
+ \sum_{p \leq \sqrt{n}} \log \left\{ \frac{n}{p} - \left( \frac{4\sqrt{n}}{\log n - 3} + \frac{p}{\log p} \right) \frac{p}{n} \log \frac{n}{p} \right\} + o(\sqrt{n})
\]

provided that

\[
\frac{n}{\log n - 3} - \frac{4\sqrt{n}}{\log n - 3} + \frac{p}{\log p} > 0 \quad \text{for} \quad p \leq n^{\frac{1}{4}}.
\]

(1)

Now for sufficiently large \( n \)

\[
\sum_{p \leq \sqrt{n}} \log \left( \frac{n}{p} \right) = \{1 + o(1)\} \left( \frac{\delta}{\log n - 3} - n^{\frac{1}{4}} \right) + o(\sqrt{n}),
\]

\[
= \{1 + o(1)\} 2(1 - \delta)(1 - \epsilon)\sqrt{n} \geq (1 - \epsilon)^2\sqrt{n};
\]

hence it remains only to show (1) and

\[
- \sum_{p \leq \sqrt{n}} \log \left\{ \frac{n}{p} - \left( \frac{4\sqrt{n}}{\log n - 3} + \frac{p}{\log p} \right) \frac{p}{n} \log \frac{n}{p} \right\} = o(\sqrt{n}).
\]

Noting that \( p \log (n/p) \) and \( p^2(1 - \log n/\log p) \) are increasing functions of \( p \) for \( p \leq \sqrt{n} \) and \( n \) sufficiently large we have

\[
\left( \frac{4\sqrt{n}}{\log n - 3} - \frac{p}{\log p} \right) p \log \frac{n}{p} = \frac{4\sqrt{n}}{\log n - 3} p \log \frac{n}{p} + p^2 \left( 1 - \frac{\log n}{\log p} \right)
\]

\[
\leq \frac{4\sqrt{n}}{\log n - 3} n^\delta (1 - \delta) \log n + n^{2\delta} \left( 1 - \frac{1}{\delta} \right)
\]

\[
= 4(1 - \delta)(1 - \epsilon)\delta n \left( \frac{2 \log n}{\log n - 3} - 1 + \epsilon \right)
\]

\[
\leq (1 - \epsilon) n(2 + \epsilon^2 - 1 + \epsilon) = (1 - \epsilon^2) n
\]

for all sufficiently large \( n \). Hence (1) holds and
\[
\sum_{p \leq n} \log \left( 1 - \left( \frac{4\sqrt{n}}{\log n - 3} - \frac{\phi}{\log \phi} \right) \frac{\phi}{n} \log \frac{n}{\phi} \right) \geq \sum_{p \leq n} 3 \log \epsilon \geq 8 \frac{\sqrt{n}}{\log n} \log \epsilon
\]

which completes the proof.

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4. The table on the preceding page lists the value of $s(n)$ for all $n \leq 113$. All entries for $s(n)$ were computed individually and checked by means of Theorem 2.

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ON THE CONTENT OF POLYNOMIALS

FRED KRAKOWSKI

1. **Introduction.** The content $C(f)$ of a polynomial $f$ with coefficients in the ring $R$ of integers of some algebraic number field $K$ is the ideal in $R$ generated by the set of coefficients of $f$. This notion plays an important part in the classical theory of algebraic numbers. Answering a question posed to the author by S. K. Stein, we show in the present note that content, as a function on $R[x]$ with values in the set $J$ of ideals of $R$, is characterized by the following three conditions:

1. $C(f)$ depends only on the set of coefficients of $f$;
2. if $f$ is a constant polynomial, say $f(x) = a$, $a \in R$, then $C(f) = (a)$, where $(a)$ denotes the principal ideal generated by $a$;
3. $C(f \cdot g) = C(f) \cdot C(g)$ (Theorem of Gauss-Kronecker, see [1, p. 105]).

2. **Characterization of content.** Denote by $[f]$ the set of nonzero coefficients of $f \in R[x]$ and call $f$, $g$ equivalent, of $f \sim g$, if $[f] = [g]$. A polynomial is said to be primitive if its coefficients are rational integers and if the g.c.d. of its coefficients is 1.

**Lemma.** Let $S$ be a set of polynomials with coefficients in $R$ and suppose it satisfies:

1. $1 \in S$;
2. if $f \in S$ and $f \sim g$, then $g \in S$;
3. if $f \cdot g \in S$, then $f \in S$ and $g \in S$.

Then $S$ contains all primitive polynomials.

Received by the editors April 27, 1964.